Equilibrium with Monotone Actions

Kyle Woodward*

September 2019

Abstract

I show that pure-strategy equilibria exist in a class of discontinuous games with private information. In my primary model actions are monotone functions on a compact and convex domain and range, and I provide conditions under which equilibria in discretizations of the primary model converge to an equilibrium in the primary model. The proof approach implies that if observable outcomes and utility are similarly continuous, they will be approximately equal in the primary model and its discretizations. I apply these results to divisible-good auctions with private information, and simultaneously prove the existence of pure strategy equilibria in discriminatory, uniform price, and hybrid formats. Outcome approximation implies that observed allocations and revenue in multi-unit auctions may be close to the theoretical predictions of divisible-good models.

*kyle.woodward@unc.edu. I would like to thank Marek Pycia for valuable advice and insights. I would also like to thank Liran Einav, Teddy Kim, Jeffrey Mensch, Moritz Meyer-ter-Vehn, Joe Ostroy, Alessandro Pavan, Phil Reny, Valentin Verdier, Bill Zame, and several generous referees, as well as seminar audiences at UCLA, Northwestern, UNC, SWET, SAET, EC'16, NC State, and INFORMS for valuable discussions and feedback. Many of the results in this paper are taken from Chapter 2 of my doctoral thesis, and were previously circulated under the title *Strategic Ironing in Pay-as-Bid Auctions*.

1 Introduction

This paper considers equilibrium existence in games with private information and discontinuous payoffs, when players' actions are monotone functions. In Bayesian games with private information, equilibrium existence can be guaranteed in settings with continuous payoffs (Athey (2001), McAdams (2003), and Reny (2011), among others), but when payoffs are potentially discontinuous in action these results do not apply. In discontinuous games, equilibrium existence is frequently established by analysis of discretizations of the model in question: when action spaces are discrete, utility is trivially continuous and equilibrium existence can be verified. Equilibrium in the discretized model is then taken to the limit as the discretization becomes fine, and best responses are verified (Simon (1987), Reny and Zamir (2004), Bagh (2010), and Kastl (2012), among others). I provide a set of conditions which formalize and unify this approach, and show that equilibrium exists in divisible-good auctions with private information.

This article's focus is on games in which actions are monotone functions. This structure is motivated by the study of divisible-good and multi-unit auction formats, in which actions are decreasing bid curves, but monotone actions are applicable to a wide variety of economic settings: any game with finite actions can be represented as a game in which actions are monotone functions, and cumulative distribution functions are also monotone.[1,2] Selection results from mathematical analysis state that a sequence of monotone functions has an almost-everywhere convergent subsequence. Then if a monotone-strategy equilibrium exists in discretizations of a game there is a natural candidate for an equilibrium of the game itself.[3] The process of constructing equilibrium in the primary model as a limit of its discretizations suggests that, when a continuum-action model is used as an approximation of a discrete real-world game, theoretical predictions should approximate their empirical counterparts.

This approach to equilibrium existence necessitates two kinds of assumptions: assumptions on discontinuities in agents' payoffs, and assumptions on behavior in discretizations of the game in question. Restrictions on the nature of payoff discontinuities enable analysis of best responses at

[1] For simplicity in the main analysis I assume that actions are monotone increasing, but this is not essential. Actions may even be nonmonotone, provided the set of local extrema is exogenous.

[2] Establishing the existence of a pure strategy Bayesian Nash equilibrium where actions are distribution functions is equivalent to establishing the existence of a mixed-strategy Bayesian Nash equilibrium.

[3] Monotonicity in this context occurs over two domains. Actions are monotone functions on the m-dimensional reals. Strategies are monotone functions from the type space to the action space.

a limiting strategy profile. Each of these restrictions has an intuitive interpretation in classical auction models. First, utility cannot be discretely reduced by a small increase in action. In many cases of interest this condition is facially violated: for example, a bidder in a single-unit auction would prefer not to win an item than to win and pay a price above her value. I therefore allow for type-dependent action spaces and provide conditions under which these constraints may be ignored. Second, an agent who obtains a discontinuous gain at a limit of opponent strategies could choose a different action and obtain nearly this gain near the limit. Payoff discontinuities in auctions typically relate to tiebreaking, and a small increase in bid can obviate the tie in a bidder's favor. Third, if an agent incurs a discontinuous loss at a limit of strategy profiles she must have an opponent who faces a gain at the same limit.[4] In private-value auctions one bidder's loss is generally another's gain.

I place restrictions on behavior in discretized models to ensure that the "limit of equilibria" is a meaningful concept. I assume that there is a sequence of discretizations, such that any action in the primary model may be arbitrarily approximated, where each discretization admits a monotone pure strategy equilibrium. As mentioned above, when actions are discrete utility is vacuously continuous in action, and equilibrium existence may be taken from the literature (Athey, 2001; McAdams, 2003; Reny, 2011). Because this paper's focus is on games with continuum actions, I take no stance on the assumptions necessary for the existence of a monotone pure-strategy equilibrium in a discrete game with private information.

Taken together, these assumptions ensure that a limit of discrete equilibria exists, and that agent utility is reasonably well-behaved at one such limit. At the limit of equilibrium strategies in the discretized models, no agent has a strategy that discontinuously improves on her limiting strategy. If she did, a nearby strategy could offer nearly as much utility, and could be approximated in a sufficiently-fine discretization. This would violate the construction of limiting strategies from a sequence of equilibrium strategies in discretized models. Furthermore, no agent's utility discontinuously falls at the limit of her equilibrium strategies in the discretized models. Some opponent would occasionally see a discontinuous improvement in ex post utility, which is ruled out in the same way as an improvement in interim utility.

[4]This requirement is conceptually similar to reciprocal upper semicontinuity (Simon, 1987), biut the precise definition is weaker.

Under these conditions utility converges with equilibrium strategies, and since no other strategy can offer a discrete improvement the limiting strategy profile is itself an equilibrium. Because many outcomes of interest are directly tied to utility, this implies that outcomes are also converging. For example, seller revenue comes from bidder payments. Then since bidder utility in equilibrium of the discretized models converges to bidder utility in equilibrium of the primary model, seller revenue is also converging. It is immediate that equilibrium in the primary model can approximate equilibrium in the discretized models, and vice versa.

I apply these results to divisible-good auctions, establishing the existence of monotone pure-strategy equilibria and probabilistic approximation of auction outcomes. While it is known that common multi-unit auction formats admit pure-strategy Bayesian Nash equilibria (McAdams, 2003; Reny, 2011), general models are viewed as intractible (Hortaçsu and Kastl, 2012) suggesting that a divisible-good approximation might yield fruitful results. In the presence of private information, equilibrium existence in divisible-good auctions has gone largely unaddressed: when goods are divisible, results which assume finite action spaces do not directly apply, and the payoff discontinuities inherent to auctions impede application of results which assume utility is continuous. Equilibrium existence in divisible-good auctions with private information is a novel contribution of this paper. I also show that, in multi-unit auctions, the distributions of quantity allocations and seller revenue are converging to their distributions in the divisible-good model. Since empirical investigations of auctions frequently address questions of efficiency and revenue, these results provide justification for application of the divisible-good model.

Generally, the results in this paper attempt to unify the approach taken toward equilibrium existence in continuum-action models. In these models it is often difficult to directly prove equilibrium existence, so equilibrium is established in nearby models and then a limit is taken (see Jackson et al. (2002), Reny (2011), among others). Viewed from this angle the results in this paper provide a set of conditions under which such methods are valid. The conditions for equilibrium existence presented in this article are similar to those found in the literature on equilibrium existence, but, in line with the proof of equilibrium existence, they place particular emphasis on behavior relative to sequences of strategies.

1.1 Related literature

This paper follows neatly from two threads of equilibrium existence literature. The first establishes (potentially mixed-strategy) equilibrium existence in models with discontinuous payoffs (as in Reny (1999)), and the second looks at the same question in models with private information and continuous payoffs (as in Athey (2001)). McAdams (2003) extends Athey's result to include multidimensional private information, and Van Zandt and Vives (2007) and Reny (2011) generalize to the case of arbitrary lattices. These results cannot be directly applied because, as is common in auction models, payoff discontinuities cannot be ruled out ex ante. Reny (1999) allows for discontinuous utility functions, but does not permit private information; his results have been extended by McLennan et al. (2011) and Barelli and Meneghel (2013).

The approach of establishing equilibrium as a limit of nearby discretized equilibria has been used by, among others, Simon (1987), Reny and Zamir (2004), Bagh (2010), and Kastl (2012). In contrast to my pure-strategy existence result under private information, Simon (1987) establishes existence in mixed strategies, without private information.[5] The limiting approach of Reny and Zamir (2004) uses a similar method to prove the existence of pure-strategy equilibria in first-price auctions; like my results here, it relies on convergence of utility, but unlike my results actions are point bids rather than generic monotone functions. Kastl (2012) provides equilibrium in distributional strategies with finite bid points, and uses this to suggest the same when bids can be arbitrary nonincreasing functions of quantity.

The condition most directly related to the ability of my conditions to extend existence results to divisible-good auctions with private information is a weakened form of reciprocal upper semicontinuity. Similar conditions have been examined by Bagh and Jofre (2006), Bagh (2010), Allison and Lepore (2014), and He and Yannelis (2016). Bagh and Jofre (2006) examines weak reciprocal upper semicontinuity, which is not necessarily satisfied by divisible-good auctions; Condition 5 requires only that most of the limiting utility can be obtained, and not that it can be dominated. Bagh (2010) employs variational convergence, which invokes dominating sequences of actions; Condition 5 can be weakened to require only that the dominating sequence of actions dominate the

[5]In later work, McAdams (2006) shows that these mixed strategies can be rendered into monotone pure strategies without affecting best-responseness. Reny (1999) shows a related result, that in a particular multi-unit auction model the mixed strategies predicted are in fact pure strategies. Other results regarding equilibrium in mixed or distributional strategies include Milgrom and Weber (1985), Kastl (2012), and He and Yannelis (2016).

original sequence (Condition 5), entirely avoiding behavior at the limit. Lastly, Allison and Lepore (2014) and He and Yannelis (2016) introduce (random) disjoint payoff matching, again requiring dominance in the limit. None of these conditions is obviously satisfied in divisible-good pay-as-bid auctions.

Relatively little is known about bidder behavior in multi-unit auctions with private information. Beyond the apparent theoretical difficulty of computing fully general revenue and efficiency rankings, progress in the analysis of parameterized models has been hampered by the inability to efficiently compute equilibrium strategies in the case where goods, as in practice, are imperfectly divisible. Meaningful results have been obtained in parameterized settings, e.g., Engelbrecht-Wiggans and Kahn (2002), Ausubel et al. (2014), Lotfi and Sarkar (2016), and Burkett and Woodward (2018).[6] Häfner (2015) demonstrates the existence of an equilibrium in distributional strategies in a discriminatory auction with constrained bids, but does not obtain a pure-strategy existence result.

Where discrete problems appear intractible, continuous approximations may offer sound and available economic insights. For example, the literature on single-unit auctions frequently employs the assumption that the set of available prices is dense. In the case of multi-unit auctions, bids may be approximated as objects determined on a dense domain of quantities, as well; there is no counterpart to this possibility in single-unit auctions, or even in combinatorial auctions. Wilson (1979) was the first to apply this approximation method in the context of multi-unit auctions, and this approximation has been used to establish results for parameterized models such as Back and Zender (1993), Wang and Zender (2002), Ausubel et al. (2014), and Pycia and Woodward (2019), but in the general case it has not even been known if an equilibrium exists. Without a sound basis for the existence of equilibrium strategies, it has been difficult to meaningfully apply the divisible-good model to policy debates.

Section 3 lays out the main results of the model. Section 4 takes these results to a class of divisible-good auctions, and proves the existence of pure-strategy equilibria as well as equilibrium approximation. Section 5 concludes.

[6] There has been work in building approximate equilibria for multi-unit auctions; see, e.g., Armantier and Sbaï (2006), Armantier et al. (2008), and Armantier and Sbaï (2009).

2 Model

The primary model is given by $\mathcal{M} = (n, u, (X_D, X_R), A, F)$; the term *primary* is used to distinguish $\mathcal{M}$ from its discretizations, defined later. There is a finite set of agents $i \in \{1, \ldots, n\}$. Agent i has private information $s_i \sim F^i$, with support $\operatorname{Supp} s_i = S \equiv (0,1)^m$. For all agents $j \neq i$, s_i is independent of s_j. Under Condition 1 below, it is without loss of generality to assume that s_i is the product of m indepdendent uniform distributions, $s_{ik} \sim \mathcal{U}(0,1)$ and for all $k' \neq k$, s_{ik} is independent of $s_{ik'}$.

Agents' actions are isotone functions from domain $X_D \subset \mathbb{R}^m$ to range $X_R \subset \mathbb{R}^m$. Both X_D and X_R are compact and convex.[7] Let Y be the set of monotone increasing functions from X_D to X_R.[8] When agent i has signal s_i, her feasible action space is $A^i(s_i) \subseteq Y$; A is the profile of feasible action spaces, $A = (A^i)_{i=1}^n$.[9] A strategy $\alpha^i : S \to Y$ is *feasible* for agent i if for all s_i, $\alpha^i(s_i) \in A^i(s_i)$. A strategy profile $\alpha = (\alpha^1, \ldots, \alpha^n)$ is feasible if α^i is feasible for all i.

Agent i's utility function is $u^i : Y^n \times S \to \mathbb{R}$; $u = (u_i)_{i=1}^n$ is the profile of utility functions. Interim expected utility for agent i is defined with respect to her own information and her opponents' strategies,

$$U^i\left(a_i, \alpha^{-i}; s_i\right) = \mathbb{E}_{s_{-i}}\left[u^i\left(a_i, \alpha^{-i}\left(s_{-i}\right); s_i\right)\middle| s_i\right].$$

Unless otherwise stated, all norms are taken to be L^1 on the relevant domain, $||a|| = \int ||a(x)||dx$ for functions and $||a|| = \sum_{i=1}^m |a_i|$ for vectors.

2.1 ε-discrete model

The proof of equilibrium existence in the primary model $\mathcal{M}$ appeals to a sequence of equilibrium strategy profiles in discretized models which approximate the primary model. The first step in this process is defining the discrete approximations.

Given a primary model $\mathcal{M}$ and $\varepsilon > 0$, an ε-approximation $\mathcal{M}^\varepsilon = (n, u, X^\varepsilon, A^\varepsilon, F)$ is derived from the primary model $\mathcal{M}$ by discretizing its feasible action spaces.

[7] The assumption that $\operatorname{Supp} s_i$, X_D, and X_R are of common dimension m is without loss of generality so long as each is of finite dimension.

[8] All results extend to the case in which different types play actions which are differently monotone, provided it is known in advance which types will employ monotone increasing actions and which will employ monotone decreasing actions. Allowing for this variation introduces additional technical overhead but little additional intuition. Similarly, nonmonotone actions can be readily accommodated, provided the set of local extrema is exogenously fixed.

[9] Type-dependent action spaces are not essential to the results but greatly simplify the exposition.

Definition 1 (Finite ε-approximation)**.** *Let $Z \subseteq \mathbb{R}^m$ be compact. $Z^\varepsilon \subseteq Z$ is a* finite ε-approximation *of Z if the following conditions hold:*

1. *Z^ε is finite;*

2. *For any $z \in Z$, there exists $\mathsf{z} \in Z^\varepsilon$ such that $||z - \mathsf{z}|| < \varepsilon$.*

Let $X^\varepsilon = (X_D^\varepsilon, X_R^\varepsilon)$, where X_D^ε and X_R^ε are finite ε-approximations of X_D and X_R, respectively. Let Y^ε be the set of isotone functions from X_R^ε to X_D^ε. For each agent i and each signal s_i, $A^{i,\varepsilon}(s_i) \subseteq Y^\varepsilon$. It need not be the case feasible actions in the discrete model are feasible in the primary model, and potentially $A^{i,\varepsilon}(s_i) \not\subseteq A^i(s_i)$.

The role of the ε-discrete model is to ensure equilibrium existence in a sequence of models approaching the primary model $\mathcal{M}$. Whether a particular discretization is appropriate is a matter of ease of satisfying the conditions set forth in Section 3; see the discussion of divisible-good auctions in Section 4.

2.2 Equilibrium

In Section 3 I take a limit of equilibrium strategies in ε-discrete models to obtian a Bayesian Nash equilibrium in the primary model $\mathcal{M}$.

Definition 2 (Constrained Bayesian Nash equilibrium)**.** *A strategy profile $(\alpha^i)_{i=1}^n$ is a* constrained Bayesian Nash equilibrium *if for all agents i,*

$$\mathbb{E}_{s_i}\left[U^i\left(\alpha^i\left(s_i\right), \alpha^{-i}; s_i\right)\right] = \sup_\alpha \mathbb{E}_{s_i}\left[U^i\left(\alpha\left(s_i\right), \alpha^{-i}; s_i\right)\right] \text{ s.t. } \alpha^i\left(s_i\right) \in A^i\left(s_i\right) \ \forall s_i.$$

A constrained Bayesian Nash equilibrium is a Bayesian Nash equilibrium in which agents are constrained to implement feasible strategies. If the action space is degenerately type-dependent, so that $A^i(s_i) = Y$ for all agents i and types s_i, a constrained Bayesian Nash equilibrium is a (standard) Bayesian Nash equilibrium.

For a strategy profile to be a constrained Bayesian Nash equilibrium it is only necessary that each agent is ex ante best responding, or is best responding with probability 1. Under an additional condition set forth in Section 3 it can be shown that there is a natural limit of ε-discrete equilibrium in which agents are certainly best responding.[10] To capture this I define a *constrained pure-strategy equilibrium* in the sense of McAdams (2003).

Definition 3 (Constrained pure-strategy equilibrium). *A strategy profile* $(\alpha^i)_{i=1}^n$ *is a* constrained pure-strategy equilibrium *if for all agents* i *and type realizations* s_i,

$$U^i\left(\alpha^i\left(s_i\right),\alpha^{-i};s_i\right)=\sup_a U^i\left(a,\alpha^{-i};s_i\right) \;\; s.t. \;\; a\in A^i\left(s_i\right).$$

As with the relation between constrained Bayesian Nash equilibrium and Bayesian Nash equilibrum, when the type-dependent action space is irrelevant, so that $A^i(s_i)=Y$ for all agents i and types s_i, a constrained pure-strategy equilibrium is a (standard) pure-strategy equilibrium.

3 Results

3.1 Conditions for equilibrium existence

The conditions used to establish equilibrium existence in the primary model $\mathcal{M}$ are mostly stated in interim utility to capture the intuition behind equilibrium existence. Most have related ex post formulations which can be easier to work with; for an example, see Lemma 11 in Appendix A.

Condition 1 (Structure of fundamentals). *For each agent* i, u^i *is bounded, and increasing and left-continuous in own signal* s_i.[11] *For all* s_i, $A^i(s_i)$ *is a complete semilattice.*

Bounded utility ensures the existence of convergent subsequences of equilibrium utilities,[12] and continuity ensures that agents who are close in type should have elements in their best responses which are near one another. The lattice structure of actions ensures that monotonicity is a meaningful concept.

Condition 2 (Imitability). *For each agent* i, $s_i<s_i'$ *implies* $A^i(s_i)\subseteq A^i(s_i')$.

A natural construction of A^i is the set of actions which generate utility above some outside option. For example, bids in a first-price auction should fall below the highest feasible value for

[10]Proving the existence of an equilibrium in which all types are best responding relies on left-continuity of utility in signal and the construction of a particular limit of ε-discrete equilibrium. Nonetheless the intuitive reasons behind the existence of the two different kinds of equilibria are fundamentally the same.

[11]A multidimensional function is left-continuous if it is coordinatewise left continuous in each argument.

[12]Where it is not of technical importance, I assume that sequences converge. Formally, all such arguments go through when applied to convergent subsequences.

the item, conditional on an agent's information. Since utility is increasing in signal, under this interpretation A^i is weakly increasing in set inclusion order. Under Condition 2, higher-type agents can always imitate lower-type agents, but not vice-versa. Condition 2 is trivially satisfied when feasible action spaces do not depend on the agent's type.

Conditions on utility

Condition 3 (Uniform upper semicontinuity). *There is a continuous function $g : \mathbb{R}_+ \times S \to \mathbb{R}_+$, $g(0; \cdot) = 0$, such that for all agents i, all types s_i, all monotone strategy profiles $(\alpha^j)_{j\neq i}$, all feasible actions $a_i \in A^i(s_i)$, and all actions $\overline{a}_i \in Y$ with $a_i \leq \overline{a}_i$,*

$$U^i\left(a_i, \alpha^{-i}; s_i\right) \leq U^i\left(\overline{a}_i, \alpha^{-i}; s_i\right) + g\left(||\overline{a}_i - a_i||\,; s_i\right).$$

Condition 3 implies that agent i's utility is upper semicontinuous in her own action, and that the modulus of semicontinuity is uniform acrosss all actions, stategy profiles, and signals. Absent type-dependent action spaces this condition is not necessarily satisfied in many cases of interest. For example, in a first price auction a bidder can receive zero utility by submitting a bid above her value but below all of her opponents' bids. If her opponents' bids are massed at her own bid, a small increase in her bid can have a disproportionately negative effect on utility. Proper construction of type-dependent action spaces plays an important role in the satisfaction of Condition 3.

Condition 4 (Local utility security). *Let $\langle(\alpha^{j,t})_{j\neq i}\rangle_{t=1}^{\infty}$ be a sequence of monotone strategies for agents $j \neq i$, converging to the strategy profile $(\alpha^{j,\star})_{j\neq i}$. For any feasible action $a_i \in A^i(s_i)$ and any $\lambda > 0$, there is a feasible action $a_i' \in A^i(s_i)$ such that $||a_i' - a_i|| \leq \lambda$, and*

$$\lim_{t\nearrow\infty} U^i\left(a_i', \alpha^{-i,t}; s_i\right) > U^i\left(a_i, \alpha^{-i,\star}; s_i\right) - \lambda.$$

Given an action for agent i and the limit of a sequence of her opponents' strategies, Condition 4 requires that there is a nearby feasible action for agent i which yields, in the limit, nearly as much utility. Local utility security is similar to payoff security, except that the sequence of opponent strategies $\langle(\alpha^{j,t})_{j\neq i}\rangle_{t=1}^{\infty}$ is not universally quantified: the specific a_i' can depend on the sequence. Following Condition 3, Condition 4 is frequently straightforward to satisfy by considering small upward deviations, $a_i' \gtrsim a_i$

Condition 5 (Limit surplus splitting)**.** *Let $\langle(\alpha^{k,t})_{k=1}^{n}\rangle_{t=1}^{\infty}$ be a sequence of monotone strategies converging to the feasible strategy profile $(\alpha^{k,\star})_{k=1}^{n}$. Suppose that there is an agent i such that*

$$\Pr{}_s\left(\lim_{t\nearrow\infty} u^i\left(\alpha^{i,t}\left(s_i\right),\alpha^{-i,t}\left(s_{-i}\right);s_i\right) > u^i\left(\alpha^{i,\star}\left(s_i\right),\alpha^{-i,\star}\left(s_{-i}\right);s_i\right)\right) > 0.$$

Then there is an agent j such that for any $\lambda > 0$ there is a sequence of feasible strategies $\langle\hat{\alpha}^{j,t}\rangle_{t=1}^{\infty}$ satisfying

1. *For all t sufficiently large, $||\hat{\alpha}^{j,t}(s_j) - \alpha^{j,t}(s_j)|| < \lambda$ for all types s_j;*
2. *With positive probability, agent j's utility improves under strategy $\hat{\alpha}^{j,t}$,*

$$\Pr{}_s\left(\lim_{t\nearrow\infty} u^j\left(\alpha^{j,t}\left(s_j\right),\alpha^{-j,t}\left(s_{-j}\right);s_j\right) < \lim_{t\nearrow\infty} u^j\left(\hat{\alpha}^{j,t}\left(s_j\right),\alpha^{-j,t}\left(s_{-j}\right);s_j\right)\right) > 0.$$

Condition 5 is related to reciprocal upper semicontinuity (Bagh and Jofre, 2006), but allows for the possibility that *all* agents' utilities face a simultaneous downward discontinuity in utility as $\alpha^t \to \alpha^\star$. Limit surplus splitting intuitively suggests that the primary model $\mathcal{M}$ has winners and losers, and a discontinuous loss in utility by one agent allows for an opponent to obtain a discontinuous gain. That the agent's opponents do not necessarily obtain a discrete increase in interim utility allows for the fact that in the limit they might also lose interim utility. It follows that Condition 5 is weaker than interim reciprocal upper semicontinuity.

Conditions on ε-discrete models

To obtain equilibrium existence I construct a sequence of equilibria in ε-discrete models; the following conditions place some restrictions on the discretization. For these conditions, let $\langle\varepsilon_t\rangle_{t=1}^{\infty}$ be a strictly decreasing sequence of real numbers converging to 0 and let $\langle\mathcal{M}^{\varepsilon_t}\rangle_{t=1}^{\infty}$ be an associated sequence of ε-discretizations of the base model $\mathcal{M}$.[13]

Condition 6 (Existence of discrete equilibrium)**.** *There is T such that, for all $t \geq T$, $\mathcal{M}^{\varepsilon_t}$ admits a monotone constrained Bayesian Nash equilibrium.*

[13] Conditions 6 and 7 can be stated as "for $\varepsilon > 0$ sufficiently small." The stated conditions offer a weaker formulation, as it is only necessary conditions are satisfied on a particular path to 0, not all paths.

Condition 6 is necessary for the proof of existence in the continuum-action case since equilibrium is constructed as a limit of equilibria of the discretized models, hence there must be equilibria in the discretized models. Satisfaction of Condition 6 can be verified with techniques from the equilibrium existence literature; see Athey (2001), McAdams (2003), and Reny (2011), among others.

Condition 7 (Approximating action spaces). *For all agents i and signals s_i:*

1. *For all $a_i \in A^i(s_i)$, there exists a monotone decreasing sequence $\langle a_i^t \rangle_{t=1}^{\infty}$ converging to a_i, such that $a_i^t \in A^{i,\varepsilon_t}(s_i)$ and $a_i^t \geq a_i$ for all t;*

2. *For all sequences $\langle a_i^t \rangle_{t=1}^{\infty}$, $a_i^t \in A^{i,\varepsilon_t}(s_i)$, and any convergent subsequence $\langle a_i^{t_k} \rangle_{k=1}^{\infty}$, there is $a_i^\star \in A^i(s_i)$ such that $a_i^{t_k} \to a_i^\star$.*

The first point of Condition 7 requires that any action in $A^i(s_i)$ can be approximated from above arbirarily closely, as the discretization becomes fine; approximation from above is practically useful when utility is uniformly upper semicontinuous (Condition 3). The second point of Condition 7 requires that actions in the discretized models cannot be too far away from actions in the continuum model; together with the first point, this can be viewed as any discretized action must approximate some action in the base model, and any action in the base model can be approximated in sufficiently fine discretized models.

Finally, I provide a condition under which the existence of constrained Bayesian Nash equilibrium implies the existence of constrained pure-strategy equilibrium, but which is not necessary to obtain the existence of a Bayesian Nash equilibrium.

Condition 8 (Type insensitivity). *Let $\underline{A}^i(s_i) = \cup_{s_i' < s_i} A^i(s_i')$. For all agents i, all opponent strategy profiles α^{-i}, all signals s_i, all $\lambda > 0$, and all $a_i \in A^i(s_i) \setminus \underline{A}^i(s_i)$, there is $a_i' \in \underline{A}^i(s_i)$ such that*

$$U^i\left(a_i', \alpha^{-i}; s_i\right) > U^i\left(a_i, \alpha^{-i}; s_i\right) - \lambda.$$

Fixing a type s_i, Condition 8 requires that restricting an agent to actions which are feasible for slightly lower types cannot discontinuously reduce her utility. When utility is left-continuous in type (Condition 1) it will be the case that types $s_i' \lesssim s_i$ can obtain roughly the same utility as type s_i. Constrained Bayesian Nash equilibrium can be translated to constrained pure-strategy equilibrium by assigning nonbest-responding types to play the supremum of lower-types' actions.

3.2 Equilibrium existence

Proofs are found in Appendix A.

I now show that the stated conditions are sufficient for the existence of a Bayesian Nash equilibrium. Assume the primary model $\mathcal{M}$ satisfies Conditions 1-5. Let $\langle \varepsilon_t \rangle_{t=1}^{\infty}$ be a strictly decreasing sequence of real numbers converging to zero, and let $\langle \mathcal{M}^{\varepsilon_t} \rangle_{t=1}^{\infty}$ be an associated sequence of ε_t-discretizations of the primary model $\mathcal{M}$ satisfying Conditions 6-7. For simplicity I will refer to the model as satisfying Conditions 1-7.

For any t, let $(\alpha^{i,t})_{i=1}^{n}$ be a monotone Bayesian-Nash equilibrium in $\mathcal{M}^{\varepsilon_t}$. Since each $\alpha^{i,t}$ is bounded, selection results (c.f. Widder (1941)) imply that there is a pointwise limit on any countable set of points.[14] It is useful that this set of points be dense, hence let $\mathcal{X}_D = X_D \cap \mathbb{Q}^m$ and $\mathcal{S} = S \cap \mathbb{Q}^m$.

Lemma 1 (Pointwise convergence on countable set)**.** *There is a strategy profile* $(\alpha^{i,\square})$ *such that for all agents* i*, all* $x \in \mathcal{X}_D$*, and all* $s \in \mathcal{S}$*,*

$$\lim_{t \nearrow \infty} \left[\alpha^{i,t}(s)\right](x) = \left[\alpha^{i,\square}(s)\right](x).$$

The sets $\mathcal{X}_D$ and $\mathcal{S}$ are countable while X_D and S are uncountable, hence the strategy profile $(\alpha^{i,\square})_{i=1}^{n}$ comprised of functions on X_D may have significant "holes." Monotone functions on compact domains are continuous almost everywhere (Lavrič, 1993), thus any monotone function that coincides with $\alpha^{i,\square}$ on $\mathcal{X}_D \times \mathcal{S}$ is L^1-equivalent to $\alpha^{i,\square}$.

Lemma 2 (Convergence to limit)**.**[15] *For all agents* i*,* $||\hat{\alpha}^i - \alpha^{i,\square}|| = 0$ *implies* $\lim_{t \nearrow \infty} ||\alpha^{i,t} - \hat{\alpha}^i|| = 0$*. Furthermore, with* s_i*-probability one,*

$$\lim_{t \nearrow \infty} \left|\left|\alpha^{i,t}(s_i) - \hat{\alpha}^i(s_i)\right|\right| = 0.$$

Any strategy which is L^1-indistinguishable from $\alpha^{i,\square}$ is a limit point of $\langle \alpha^{i,t} \rangle_{t=1}^{\infty}$. There is freedom in the precise limit taken, permitting the construction of *supremum-limit strategies* $\overline{\alpha}^i$.

[14] In a single dimension, Helly's selection theorem guarantees that any sequence of bounded monotone functions on a compact domain admits a convergent subsequence. In multiple dimensions these results appeal to total boundedness, which is not exogenously guaranteed in many game theoretic models. Instead, pointwise convergence and monotonicity are used to derive L^1 convergence.

[15] In Appendix A this is proved as Lemmas 7 and 8.

These are strategies at which each type realization s_i is playing an action that is the least upper bound of actions for all lower type realizations $s_i' < s_i$. Condition 1 requires that utility is left-continuous in signal, thus an action which is the supremum of lower types' best responses is a natural action to examine as a type's own best response.

Definition 4 (Supremum-limit strategy). *$\overline{\alpha}^i$ is a* supremum-limit *strategy for agent i if for all $s_i \in S$,*

$$\overline{\alpha}^i (s_i) = \sup_{s_i' < s_i} \alpha^{i,\square} (s_i') .$$

As strategies converge, so too does utility for almost all agents. This is the bulk of the proof of equilibrium existence.

Lemma 3 (Utility convergence almost everywhere). *For all agents i,*

$$\Pr{}_{s_i} \left(\lim_{t \nearrow \infty} U^i \left(\alpha^{i,t} (s_i) , \alpha^{-i,t}; s_i \right) \neq U^i \left(\overline{\alpha}^i (s_i) , \overline{\alpha}^{-i}; s_i \right) \right) = 0.$$

Convergence of utility is established by separately showing that utility cannot jump up at the limit, nor can it jump down at the limit, with positive probability. A rough sketch of the proof highlights the role played by each condition.

Suppose an agent's interim utility discontinuously improves at the limit. Local utility security (Condition 4) implies that there is a constant action such that the agent can obtain nearly the utility at the limit of all strategies, in the limit of opponent strategies. Discretized action spaces can approximate actions from above (Condition 7), and uniform upper semicontinuity (Condition 3) implies that these approximations cannot give utility discontinuously worse than the constant action, which improved over the limit. Then if utility jumps up at the limit, there are actions in the discretizations that improve the agent's interim utility, contradicting the construction of the sequence from equilibrium strategy profiles.

Suppose instead that an agent's interim utility discontinuously falls at the limit. Limit surplus splitting (Condition 5) implies that there is an opponent whose ex post utility, with positive probability, discontinuously increases at the limit. Essentially the same arguments from the case where the agent's interim utility discontinuously improves hold in this case for the agent's opponent, being careful to account for the fact that the opponent's interim utility may not be discontinuously improving at the limit. Again, the construction of the sequence from equilibrium strategy profiles is contradicted.

Returning to Condition 4 gives that supremum-limit strategies are mutual best responses.

Theorem 1 (Constrained Bayesian Nash equilibrium)**.** *Suppose that Conditions 1-7 are satisfied. Then the supremum-limit strategy profile* $(\overline{\alpha}^i)_{i=1}^n$ *forms a monotone constrained Bayesian Nash equilibrium in the model* $\mathcal{M}$*. For all agents* i*,*

$$\mathbb{E}_{s_i}\left[U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right)\right] \geq \sup_{a_i \in A^i(s_i)} \mathbb{E}_{s_i}\left[U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right)\right].$$

Corollary 1 (Symmetric equilibrium)**.** *Suppose that Conditions 1-7 are satisfied. If* $A^i = \hat{A}$ *and* $u^i = \hat{u}$ *for all agents* i*, then* $\mathcal{M}$ *admits a symmetric monotone constrained Bayesian Nash equilibrium* $(\hat{\alpha})_{i=1}^n$*.*

If agent i has an action a_i which discretely improves on $\overline{\alpha}^i(s_i)$, she has an action close to a_i which is an improvement over some $\alpha^{i,t}$ against $(\alpha^{j,t})_{j\neq i}$. For ε_t sufficiently small, a_i can be approximated into $A^{i,\varepsilon_t}(s_i)$ at a loss that is of order $g(C\varepsilon_t; s_i)$. Since $(\alpha^{j,t})_{j=1}^n$ is a Bayesian Nash equilibrium, in which almost all signal realizations are best-responding, this is a contradiction if a positive mass of agents have utility-improving actions.

Supremum-limit strategies are useful but not necessary to obtain a Bayesian Nash equilibrium as the limit of discrete equilibria. Lemma 2 establishes that actions are converging with probability one (with respect to type realization), and the proof of Lemma 3 can be adapted to show that at all such type realizations agents are best-responding. However, the construction of a constrained pure-strategy equilibrium requires the satisfaction of Condition 8 and the construction of supremum-limit strategies.

Theorem 2 (Constrained pure-strategy equilibrium)**.** *Let* $(\overline{\alpha}^i)_{i=1}^n$ *be a monotone constrained Bayesian Nash equilibrium in supremum-limit strategies. If Condition 8 is satisfied, the strategy profile* $(\overline{\alpha}^i)_{i=1}^n$ *is a monotone constrained pure-strategy equilibrium: for each agent* i *and all signal realizations* s_i*,*

$$U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) \geq \sup_{a_i \in A^i(s_i)} U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right).$$

Finally, the preceding results establish existence of constrained equilibrium, in which actions must be feasible. With a further condition on the nature of type-dependent action spaces, equilibrium existence obtains in the derived model in which $A^i \equiv Y$ for all agents i, and an equilibrium with type-independent action spaces is equivalent to a standard equilibrium.

Theorem 3 (Unconstrained equilibrium existence)**.** *Suppose that for all $y \in Y$, all opponent strategy profiles α^{-i}, and all $\lambda > 0$, there exists $a_i \in A^i(s_i)$ such that*

$$U^i\left(a_i, \alpha^{-i}; s_i\right) > U^i\left(y, \alpha^{-i}; s_i\right) - \lambda.$$

Let $(\overline{\alpha}^i)_{i=1}^n$ be a monotone constrained pure-strategy equilibrium in supremum-limit strategies of the model $\mathcal{M}$. Then the strategy profile $(\overline{\alpha}^i)_{i=1}^n$ is a monotone constrained pure-strategy equilibrium in the model $\mathcal{M}^Y = (n, u, X, A^Y, F)$, where $A^{Yi}(s_i) = Y$.

3.3 Equilibrium approximation

The construction of equilibrium in $\mathcal{M}$ as a profile supremum-limit strategies suggests that equilibrium in $\mathcal{M}$ may be near equilibrium in the ε_t-discretization $\mathcal{M}^{\varepsilon_t}$.[16] In these results, I assume that $\langle(\alpha^{i,t})_{i=1}^n\rangle_{t=1}^\infty$ is a sequence of monotone constrained Bayesian Nash equilibria of the ε_t-discrete models $\mathcal{M}^{\varepsilon_t}$ converging to the supremum-limit strategy profile $(\overline{\alpha}^i)_{i=1}^n$.

Definition 5 (Utility-relevant function)**.** *Let (W, T_W) be a topological space. A function $w : Y^n \to W$ is* utility-relevant *if for any convergent sequence of strategy profiles, $\langle\alpha^t\rangle_{t=1}^\infty \to \alpha^\star$, $w(\alpha^t(s)) \not\to w(\alpha^\star(s))$ implies that there is an agent i such that $u^i(\alpha^t(s); s_i) \not\to u^i(\alpha^\star(s); s_i)$.*

A utility-relevant function is a mapping from actions to a set W such that its own discontinuities imply discontinuities in some agent's utility. For example, in many auction models quantity allocations are utility relevant: a discontinuous change in utility in general represents a discontinuous change in quantity.[17]

[16] In models in which equilibrum is unique this approximation is strict, in the sense that all equilibria converge to the unique equilibrium in $\mathcal{M}$. Without uniqueness the strongest statement possible is, "The sequence of equilibria in $\mathcal{M}^{\varepsilon_t}$ contains a subsequence which converges to an equilibrium of $\mathcal{M}$."

[17] In auction models without private information it is straightforward to construct sequences of actions at the limit of which quantity is discontinuous but utility is not (for related examples, see Reny (1999) and Jackson et al. (2002), among others). With massless private information and strictly monotone private values, these constructions go away.

Theorem 4 (Equilibrium approximation)**.** *Let* (W, T_W) *be a topological space, and suppose that* $w : Y^n \to W$ *is utility-relevant. Then for almost all type profiles* s*,*

$$\lim_{t \nearrow \infty} w\left(\alpha^t(s)\right) = w\left(\overline{\alpha}(s)\right).$$

Proof. This is an immediate consequence of utility-relevance of w and the construction of $\overline{\alpha}^i$ as a limit of $\alpha^{i,t}$ at which almost all utilities converge. □

Corollary 2 (Probabilistic approximation)**.** *Let* $w : Y^n \to W$ *be utility-relevant. Then*

$$w\left(\alpha^t(s)\right) \xrightarrow{P} w\left(\overline{\alpha}(s)\right).$$

Corollary 2 establishes that the base model may be empirically near its discretizations. Since models in which the underlying action space is a continuum — or, in this case, mappings from one continuum to another — are frequently meant as approximations of discrete real-world applications, Corollary 2 suggests that such models may be empirically useful; in Section 4 I use this result to show that in common multi-unit auction formats equilibrium allocations and revenues converge.

It is straightforward extend the model to explicitly include exogenous independent randomness Z, such as might be necessary for an anonymous tiebreaking rule in an auction. Theorem 4 and Corollary 2 naturally extend to this setting, when utility relevance is adjusted to account for the exogenous randomness Z. This extension necessitates additional notation without offering additional insight, so I do not pursue it here.

4 Application: divisible-good auctions

I now apply the equilibrium existence results from Section 3 to prove that equilibrium exists in divisible-good auctions with private information. There are $n \geq 2$ bidders, $i \in \{1, \ldots, n\}$, participating in an auction for $\hat{Q}$ units of a perfectly divisible commodity, where $\hat{Q}$ is determined by a random realization $z_Q \sim \mathcal{U}(0,1)$ and $\operatorname{Supp} \hat{Q} \subset [0, \bar{Q}]$. Bidder i's type $s_i \sim \mathcal{U}(0,1)$ is private information, and for all agents $j \neq i$, s_i and s_j are independent. Bidder i has marginal value function $v^i : [0, \bar{Q}] \times (0,1) \to \mathbb{R}_+$, where $v^i(q; s_i)$ is her marginal value for quantity q when her type is s_i. v^i

is bounded, decreasing in q, and strictly increasing and continuous in s_i.

Bidders compete for shares of the aggregate quantity $\hat{Q}$. Bidder i submits a weakly positive, weakly decreasing bid function b_i to the auctioneer, expressing her marginal willingness to pay for quantity q. Bidder i's bidding strategy is β^i, so that when her type is s_i she submits bid function $b_i = \beta^i(\cdot; s_i)$. I will denote the bidder's implicit demand functions by $\overline{\varphi}^i$ and $\underline{\varphi}^i$,

$$\overline{\varphi}^i (p; s_i) = \sup \left\{ q : \ \beta^i (q; s_i) \geq p \right\}, \quad \underline{\varphi}^i (p; s_i) = \inf \left\{ q : \ \beta^i (q; s_i) \leq p \right\}.^{18}$$

Conditional on the random shock z_Q, the auctioneer aggregates the submitted bid functions and computes the market-clearing price $p^\star$,

$$p^\star = \inf \left\{ p : \sum_{i=1}^{n} \underline{\varphi}^i (p; s_i) \leq \hat{Q} \leq \sum_{i=1}^{n} \overline{\varphi}^i (p; s_i) \right\}.^{19}$$

Given this price, the auctioneer allocates to each agent her demand at this price. If $\underline{\varphi}^i(p^\star; s_i) = \overline{\varphi}^i(p^\star; s_i)$ for all i, then $q^i(s_1, \ldots, s_n) = \underline{\varphi}^i(p^\star; s_i)$. Otherwise, the auctioneer employs a random priority tiebreaking rule.[20] Let z_q be a random permutation of agents $\{1, \ldots, n\}$, let $\iota(i)$ be such that $z_{q\iota(i)} = i$, and let $T(p) = \sum_{i=1}^{n} \overline{\varphi}^i(p; s_i) - \underline{\varphi}^i(p; s_i)$ be rationable demand at p. Bidder i's allocation is

$$q^i \left(b^i, b^{-i}; p, z_q\right) = \underline{\varphi}^i (p; s_i) + \min \left\{ T(p) - \sum_{\iota(k) < \iota(i)} \overline{\varphi}^k (p; s_k) - \underline{\varphi}^k (p; s_k), \overline{\varphi}^i (p; s_i) - \underline{\varphi}^i (p; s_i) \right\}_+ .$$

Henceforth let $z = (z_Q, z_q)$.

Once allocations are determined the auctioneer computes transfers from each of the bidders. To capture many common auction formats I define a *standard transfer rule*.

Definition 6 (Standard transfer rule)**.** *Let Y be the set of monotone decreasing functions from $[0, \bar{Q}] \to \mathbb{R}_+$. The transfer rule $\tau^i : [0, \bar{Q}] \times Y \times \mathbb{R}_+ \times Y^{n-1} \times SuppZ$ is* standard *if*

[18] If there is no q such that $\beta^i(q; s_i) \geq p$, then $\overline{\varphi}^i(p; s) = 0$, and if there is no q such that $\beta^i(q; s_i) \leq p$, then $\underline{\varphi}^i(p; s) = \bar{Q}$. Because bids are defined only on the domain of available quantities, $\overline{\varphi}^i(0; s_i) = \bar{Q}$. That $\overline{\varphi}^i(0; s_i) = \bar{Q}$ for all i and all s ensures that all acceptable bid functions will generate well-defined market outcomes.

[19] Although $p^\star$ is a function of $(\beta^i)_{i=1}^n$ and z_Q, for simplicity of notation I write it as its own random variable. The dependence of $p^\star$ on its inputs will be treated properly where necessary.

[20] As noted in Häfner (2015) and elsewhere, the tiebreaking rule is not essential to the existence of a pure-strategy equilibrium in the multi-unit discretization of the divisible good model. Corollary 5 shows that this carries over to the divisible-good model itself.

1. $\tau^i \equiv \tau$ *is symmetric across agents;*

2. τ *is increasing and uniformly continuous in the bidder's allocation* q*, the bidder's submitted demand* b_i*, the market-clearing price* p*, and opponent bids* b_{-i}*;*

3. $d^+\tau/dq \in [p, b_i(q)]$ *for all* q *such that* $b_i(q) > p$*;*

4. *Interim expected transfers* $\mathbb{E}_{s_{-i}}[\tau(q_i; b_i, p, b_{-i}, z)]$ *are submodular in bid.*

Remark 1. *Many common auction formats employ standard transfer rules:*

- *When* $\tau(q; b_i, p, b_{-i}, z) = \int_0^q b_i(x)dx$*, the mechanism is a discriminatory auction;*
- *When* $\tau(q; b_i, p, b_{-i}, z) = pq$*, the mechanism is a uniform-price auction;*[21]
- *When* $\tau(q; b_i, p, b_{-i}, z) = \lambda \int_0^q b_i(x)dx + (1-\lambda)pq$*, the mechanism is a hybrid auction;*
- *When* $\tau(q; b_i, p, b_{-i}, z) = \bar{p}^\alpha \bar{q} + \int_{\bar{q}}^q b_i(x)dx$*, and* $\bar{p}^\alpha$ *is the* α *bid percentile and* $\bar{q} = \underline{\varphi}^i(\bar{p}^\alpha)$*, the mechanism is a quantile-hybrid auction.*

Given the transfer rule t, bidder i's ex post utility is

$$u^i(b_i, b_{-i}; s_i) = \mathbb{E}_z \left[\int_0^{q^i(b_i, b_{-i}; z)} v^i(x; s_i)\, dx - \tau\left(q^i(b_i, b_{-i}; p^\star, z); b_i, p^\star, b_{-i}, z\right) \right].$$

Translating the divisible-good auction to the existence model, let $X_D = [0, \bar{Q}]$ and $X_R = [0, \bar{b}]$, where $\bar{b} > \max_i \sup_{s_i} v^i(0; s_i)$. Let $Y^\gamma \subset Y$ be the set functions in Y which are Lipschitz continuous with modulus γ. For an agent i with type s_i, the feasible action space is

$$A^i(s_i) = \left\{ y \in Y^\gamma : \; y \leq v^i(\cdot; s_i) \right\}.^{22}$$

[21] In divisible-good auctions the market price p is perfectly recoverable from b_i, b_{-i}, and z. If p is omitted as an argument to the transfer rule, the uniform-price auction is non-standard since payments are not uniformly continuous in bid — a small change in submitted bids can dramatically affect the market clearing price. A standard transfer rule must have a representation which is uniformly continuous in its parameters, but this representation does not need to be unique.

[22] Kastl (2012) gives a model in which bidders in a uniform-price auction submit bids that are occasionally above their value functions. Similarly, Bertrand competition without private information involves submitting a constant bid "to infinity." The results here establish the existence of an equilibrium in which all bidders submit bids weakly below their value functions, but do not claim that all equilibria must exhibit this property. In related work, Pycia and Woodward (2019) show in a model without private information that all relevant bids must be weakly below values.

Since values are weakly decreasing in quantity and $v^i(0; s_i) < \bar{b}$ for all agents i and type realizations s_i, $A^i(s_i)$ contains all weakly positive, Lipschitz γ-continuous decreasing functions that are bounded above by the agent's true marginal value.[23]

Lemma 4 (Current transfer continuous in bid)**.** *Let $\varepsilon > 0$, and let $b \in A^i(s_i)$ and $b' \in Y$ be such that $||b - b'|| < \varepsilon$. Then there is $\lambda > 0$ such that for any q, any z, and any bid profile b_{-i} of bidder i's opponents,*

$$\left|\tau\left(q; b, p^\star, b_{-i}, z\right) - \tau\left(q; b', p^{\star\prime}, b_{-i}, z\right)\right| < \lambda.$$

Proof. By construction, $||b - b'|| < \varepsilon$. Since bidder i's allocation must weakly increase, the effect on price can be bounded by $|b(q_i) - b'(q_i)|$. Because bids are Lipschitz γ-continuous, it must be that $|b(q_i) - b'(q_i)| \leq \sqrt{2\gamma\varepsilon}$. Uniform continuity of τ in its arguments completes the proof. □

I prove the existence of a pure-strategy equilibrium by verifying the conditions necessary to apply Theorem 3. Conditions 1 and 2, on the structure of utility and the type-dependent action spaces, are straightforward to demonstrate, and are omitted. Detailed proofs of the following results are given in Appendix B.

Condition 3 (Uniform upper semicontinuity)**.** Quantity allocations are monotone in bid, and a slight increase in bid from b to b' will never negatively affect quantity. If bid b' yields strictly greater quantity, since the original bid b was below the agent's marginal value $v^i(\cdot; s_i)$, b' can be only slightly above the agent's marginal value function $v^i(\cdot; s_i)$, and any gross utility loss from additional quantity is small. Lemma 4 gives that transfers for units already won are continuous in bid. Taken together, this implies that upward deviations cannot be discretely harmful.

Condition 4 (Local utility security)**.** Consider any feasible bid function b_i and $\lambda > 0$. When bidder i increases her bid from b_i to $b_i + \lambda$, bounded where appropriate by $v^i(\cdot; s_i)$, uniform upper semicontinuity implies that bidder i's utility in the limit is not discretely worse than her utility at the limit. As opponents' bids converge, a discrete upward shift in bidder i's bid will yield a weak increase in the quantity she is allocated. If this is not true at the limit, the discrete gap in bids implies that her opponents' actions are not converging. Since the upward shift is near a feasible

[23] Lipschitz continuity of bids is inessential and is used to ensure that the the market-clearing price $p^\star$ is well-behaved in bids; it is eliminated in Proposition 1. When t can be written indepedent of the market-clearing price — as in the discriminatory and quantile-hybrid auctions — the Lipschitz constraint can be ignored. It is retained so that existence can be proved simultaneously in all auctions with standard transfers.

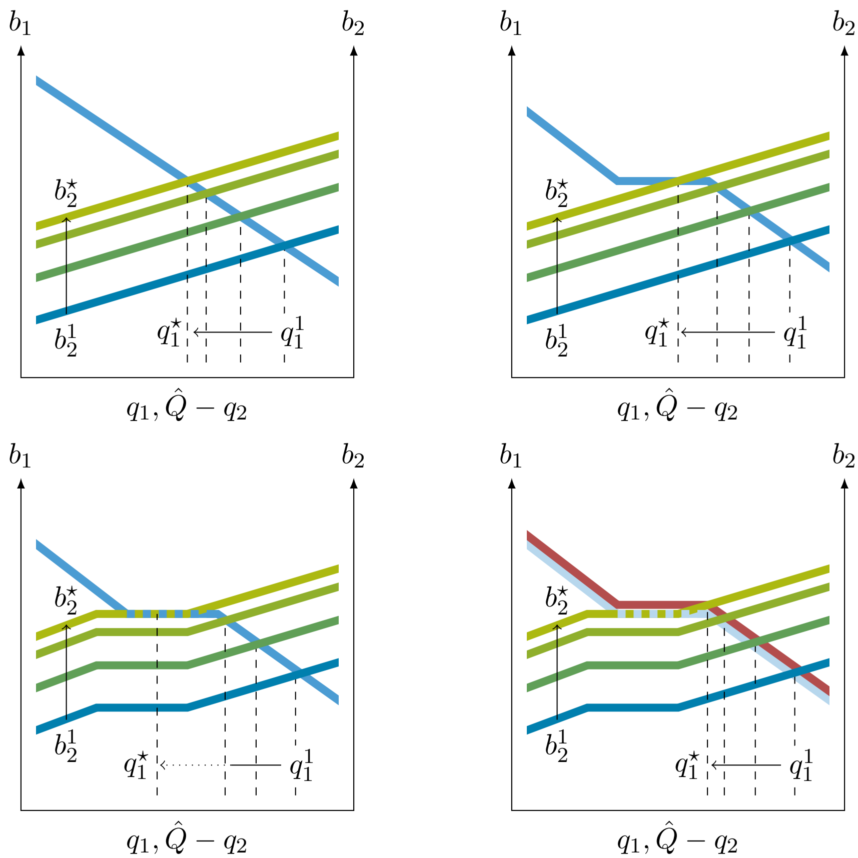

Figure 1: If agent i's utility does not converge in the limit of agent j's actions to her utility at the limit of agent j's actions, it must be that quantity is not converging; if quantity is not converging, it must be that bids are equal and along a common flat. On this interval the tiebreaking rule must be employed, hence a small upward deviation will yield a discretely greater allocation; this deviation is feasible by the assumption that utility is not converging.

action profile, any losses caused by increasing her bid are commensurate to the size of the shift. This intuition is illustrated in Figure 1.

Condition 5 (Limit surplus splitting)**.** Surplus splitting is implied by market clearing. In particular, i's utility can jump down at the limit only if her allocated quantity jumps down or her transfer jumps up. Lemma 4 rules out the latter the case. If quantity allocated jumps down with positive probability, then by market clearing there is an opponent who, with positive probability, witnesses a discrete quantity increase at the limit. Since bids are bounded above by values, the opponent's utility will increase. This is shown in Lemma 22.

Condition 8 (Type insensitivity)**.** If a particular type s_i has a best response that is not available to lower type realizations, continuity of marginal value in type and the upper bound on the feasible action space imply that the best response is occasionally equal to her marginal value. Consider an alternate bid function that is at least λ' below the bidder's marginal value. Since standard transfers are monotone in own bid, this will weakly decrease the bidder's payment for any allocation; however, it may also reduce the quantity she receives. If this slight reduction in bid reduces her allocation, her best response must be close to her true marginal value function. The lost quantity does not result in much lost utility, and the bidder can achieve most of her maximum utility by constraining her bid to be within lower types' feasible action spaces.

The remaining conditions for constrained equilibrium existence require an appropriate sequence of ε-discrete models. For this, let $\langle \varepsilon_t \rangle_{t=1}^{\infty}$ be a decreasing sequence converging to zero, and for any $\varepsilon > 0$ let X_D^{ε} and X_R^{ε} be given by

$$X_D^{\varepsilon} = \mathbb{Z}\varepsilon \cap X_D, \quad X_R^{\varepsilon} = \mathbb{Z}\varepsilon^2 \cap X_R.$$

Let Y^{ε} be the set of decreasing functions from X_D^{ε} to X_R^{ε}. For any s_i, let $A^{i,\varepsilon}(s_i)$ be given by

$$A^{i,\varepsilon}(s_i) = \left\{ y \in Y^{\varepsilon} : \ y(x) - y(x+\varepsilon) \in (0, \gamma\varepsilon], \text{ and } v^i(k\varepsilon; s_i) > 0 \implies y(k\varepsilon) \leq v^i(k\varepsilon; s_i) + \varepsilon^2 \right\}.$$

That is, the set of feasible bid functions is the set of strictly decreasing and Lipschitz γ-continuous functions on X_D^{ε} that are not too far above values (on X_D^{ε}). Equilibrium existence is assured in discrete discriminatory and uniform-price auctions, even without the Lipschitz constraint (Reny, 2011). Lipschitz continuity unifies the proof of equilibrium existence across all divisible-good auctions with common transfers, and does not affect equilibrium existence. Appendix B gives formal proofs that this ε-discretized model satisfies the appropriate conditions.

Lemma 5 (Constrained equilibrium existence in divisible-good auctions)**.** *When transfers are standard and bids are Lipschitz continuous and weakly below values, divisible-good auctions with private information admit monotone constrained pure-strategy equilibria.*

Type-dependent action spaces can be relaxed. If a bid function is infeasible, it is somewhere above the bidder's true marginal value. Consider an alternative bid function, shifted upward slightly from the original and bounded above by the bidder's true value; this alternative bid function is feasible. Vertical shifts cannot discontinuously affect the market-clearing price, so the transfer to the auctioneer varies continuously in this deviation, holding quantity fixed. Then if this deviation is discontinuously unprofitable it must be that quantity is falling, and for quantity to fall discontinuously it must be as a result of bounding the bid function above by marginal value. Anytime a bid above marginal value determines the quantity allocation the market price is above marginal value, and under a standard transfer rule the marginal payment for this unit is above its marginal value. It follows that the alternative bid cannot be discontinuously unprofitable. Then the antecedent of Theorem 3 is satisfied, and there exists an equilibrium in the auction model without type-dependent action spaces.

Corollary 3 (Lipschitz equilibrium existence in divisible-good auctions)**.** *When transfers are standard and bids are Lipschitz continuous, divisible-good auctions with private information admit monotone pure-strategy equilibria.*

The approach taken in Section 3 to establish equilibrium existence applies equally well to convergence as the Lipschitz modulus approaches infinity. Bids and aggregate demand converge, and for any signal profile the market price either converges or jumps discontinuously downward. If all terms are converging, that limiting strategies constitute an equilibrium follows the same argument used to show that the limit of discrete equilibria is an equilibrium. If market price jumps down at the limit, either no agent's utility is affected (as in a discriminatory auction) or some agent sees a discrete utility improvement at the limit. But as in earlier arguments this agent could have realized this utility improvement near the limit, contradicting the limit being constructed from a sequence of equilibria.

Proposition 1 (Equilibrium existence in divisible-good auctions)**.** *Divisible-good auctions with standard transfers and private information admit monotone pure-strategy equilibria.*

As concerns market outcomes, it is straightforward to show that allocations are utility-relevant: a discrete shift in allocation is associated with a discrete loss of per-unit margins. This immediately implies that seller revenues are also utility-relevant. Theorem 2 then implies that quantity and revenue in the ε_t-discrete auctions are approximated by quantity and revenue in the divisible-good auction.

Corollary 4 (Probabilistic convergence of observables)**.** *Let $q : Y \to \mathbb{R}_+^n$ and $\pi : Y \to \mathbb{R}_+$ represent ex post expected allocations and revenue, respectively, in the divisible-good model $\mathcal{M}$. If $\langle \beta^t \rangle_{t=1}^{\infty}$ is a* *sequence of monotone pure-strategy equilibria in the ε_t-discrete models converging to the supremum-limit strategy profile $\overline{\beta}$, then*

$$q\left(\beta^t(s)\right) \xrightarrow{P} q\left(\overline{\beta}(s)\right) \text{ and } \pi\left(\beta^t(s)\right) \xrightarrow{P} \pi\left(\overline{\beta}(s)\right).$$

5 Conclusion

This article proves the existence of monotone Bayesian Nash and pure-strategy equilibria in games in which actions are monotone functions. Application of these results to an economic model requires the specification of type-dependent action spaces and a sequence of discretized models such that (Condition 3) slight upward deviations are not discretely harmful, (Condition 4) utility at a limit of actions can be nearly achieved in a limit of actions, (Condition 5) one agent's loss can be transformed into another agent's gain, and (Condition 6) the discretized models each admit a monotone Bayesian Nash equilibrium. Equilibrium existence is established by examining a limit of a sequence of equilibria of discretized models, and showing that utility must converge. This result immediately suggests that equilibrium (and equilibrium outcomes) in the primary model can provide an approximation of equilibria in the nearby discrete games.

I apply these results to a model of divisible-good auctions with private information, under constraints on the transfers made to the auctioneer. These models satisfy the conditions set forth for equilibrium existence, allowing me to establish equilibrium existence in a broad class of divisible-good auctions. The equilibrium approximation results show that quantity allocations and seller revenue in the divisible-good model are close to their counterparts in nearby multi-unit auctions. This suggests that the divisible-good auction model could be a fruitful approach to understanding multi-unit auctions, which are currently believed to be intractable.

References

B. A. Allison and J. J. Lepore. Verifying payoff security in the mixed extension of discontinuous games. *Journal of Economic Theory*, 152:291–303, 2014.

O. Armantier and E. Sbaï. Estimation and comparison of treasury auction formats when bidders are asymmetric. *Journal of Applied Econometrics*, 21(6):745–779, 2006.

O. Armantier and E. Sbaï. Comparison of alternative payment mechanisms for French Treasury auctions. *Annals of Economics and Statistics/Annales d'Économie et de Statistique*, pages 135–160, 2009.

O. Armantier, J.-P. Florens, and J.-F. Richard. Approximation of Nash equilibria in Bayesian games. *Journal of Applied Econometrics*, 23(7):965–981, 2008.

S. Athey. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica*, 69(4):861–889, 2001.

L. M. Ausubel, P. Cramton, M. Pycia, M. Rostek, and M. Weretka. Demand reduction and inefficiency in multi-unit auctions. *The Review of Economic Studies*, 81(4):1366–1400, 2014.

K. Back and J. F. Zender. Auctions of divisible goods: On the rationale for the Treasury experiment. *Review of Financial Studies*, 6(4):733–764, 1993.

A. Bagh. Variational convergence: approximation and existence of equilibria in discontinuous games. *Journal of Economic Theory*, 145(3):1244–1268, 2010.

A. Bagh and A. Jofre. Reciprocal upper semicontinuity and better reply secure games: a comment. *Econometrica*, 74(6):1715–1721, 2006.

P. Barelli and I. Meneghel. A note on the equilibrium existence problem in discontinuous games. *Econometrica*, 81(2):813–824, 2013.

J. Burkett and K. Woodward. Uniform-price auctions with a last accepted bid pricing rule. 2018. Working paper.

R. Engelbrecht-Wiggans and C. M. Kahn. Multiunit auctions in which almost every bid wins. *Southern Economic Journal*, pages 617–631, 2002.

S. Häfner. Value bounds and best response violations in discriminatory share auctions. 2015.

W. He and N. C. Yannelis. Existence of equilibria in discontinuous bayesian games. *Journal of Economic Theory*, 162:181–194, 2016.

A. Hortaçsu and J. Kastl. Valuing dealers' informational advantage: A study of Canadian Treasury auctions. *Econometrica*, 80(6):2511–2542, 2012.

M. O. Jackson, L. K. Simon, J. M. Swinkels, and W. R. Zame. Communication and equilibrium in discontinuous games of incomplete information. *Econometrica*, 70(5):1711–1740, 2002.

J. Kastl. On the properties of equilibria in private value divisible good auctions with constrained bidding. *Journal of Mathematical Economics*, 48(6):339–352, 2012.

B. Lavrič. Continuity of monotone functions. *Archivum Mathematicum*, 29(1):1–4, 1993.

M. H. Lotfi and S. Sarkar. Uncertain price competition in a duopoly with heterogeneous availability. *IEEE Transactions on Automatic Control*, 61(4):1010–1025, 2016.

D. McAdams. Isotone equilibrium in games of incomplete information. *Econometrica*, 71(4):1191–1214, 2003.

D. McAdams. Monotone equilibrium in multi-unit auctions. *The Review of Economic Studies*, 73 (4):1039–1056, 2006.

A. McLennan, P. K. Monteiro, and R. Tourky. Games with discontinuous payoffs: A strengthening of Reny's existence theorem. *Econometrica*, 79(5):1643–1664, 2011.

P. R. Milgrom and R. J. Weber. Distributional strategies for games with incomplete information. *Mathematics of Operations Research*, 10(4):619–632, 1985.

M. Pycia and K. Woodward. Auctions of homogeneous goods: A case for pay-as-bid. Working paper, 2019.

P. J. Reny. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica*, 67(5):1029–1056, 1999.

P. J. Reny. On the existence of monotone pure-strategy equilibria in Bayesian games. *Econometrica*, 79(2):499–553, 2011.

P. J. Reny and S. Zamir. On the existence of pure strategy monotone equilibria in asymmetric first-price auctions. *Econometrica*, 72(4):1105–1125, 2004.

L. K. Simon. Games with discontinuous payoffs. *The Review of Economic Studies*, 54(4):569–597, 1987.

T. Van Zandt and X. Vives. Monotone equilibria in bayesian games of strategic complementarities. *Journal of Economic Theory*, 134(1):339–360, 2007.

J. J. Wang and J. F. Zender. Auctioning divisible goods. *Economic Theory*, 19(4):673–705, 2002.

D. V. Widder. *The Laplace Transform.* Princeton University Press, 1941.

R. Wilson. Auctions of shares. *The Quarterly Journal of Economics*, pages 675–689, 1979.

A Proofs of main results

The proof of equilibrium existence proceeds by defining *limiting strategies*, derived from equilibria of a sequence of ε_t-discrete models $\mathcal{M}^{\varepsilon_t}$. Because the proofs below frequently consider sets constrained to the rational numbers, the following shorthands from the main text are useful:

$$\mathcal{S} \equiv S \cap \mathbb{Q}^m, \;\; \mathcal{S}^C \equiv S \setminus \mathcal{S}; \quad\; \mathcal{X} \equiv X_D \cap \mathbb{Q}^m, \;\; \mathcal{X}^C \equiv X_D \setminus \mathcal{Q}^m.$$

Definition 7 (Limiting strategies)**.** *Strategies* $(\alpha^{i,\square})_{i=1}^n$ *are* limiting strategies *if there exists a monotone decreasing sequence* $\langle \varepsilon_t \rangle_{t=1}^\infty$*,* $\varepsilon_t \searrow 0$*, and a sequence of equilibria of the* ε_t*-discrete model* $\mathcal{M}^{\varepsilon_t}$*,* $\langle (\alpha^{i,t})_{i=1}^n \rangle_{t=1}^\infty$ *such that:*

1. $\alpha^{i,\square}$ *is monotone in all arguments;*

2. *For all* $(x, s_i) \in \mathcal{X} \times \mathcal{S}$*,* $[\alpha^{i,t}(s_i)](x) \to [\alpha^{i,\square}(s_i)](x)$*.*

At all rational coordinate pairs, limiting strategies take values equal to the limits of equilibrium strategies in the ε_t-discrete models at these points. When either coordinate is irrational, limiting strategies may take any value which satisfies the stated monotonicity constraints. Monotonicity of $\alpha^{i,\square}(s_i)$, as stated in point 1 above, is guaranteed by monotonicity of functions in $A^i(s_i)$, however monotonicity of $[\alpha^{i,\square}(\cdot)](x)$ must be explicitly stated: although the existence results in Reny (2011) guarantee the existence of a monotone equilibrium in $\mathcal{M}^{\varepsilon_t}$, it is possible that in some contexts a nonmonotone equilibrium will exist. The proof of existence below assumes that both strategies and actions are monotone, hence point 1 is crucial.

Lemma 6 (Existence of limiting strategies)**.** *Given any monotone decreasing sequence* $\langle \varepsilon_t \rangle_{t=1}^{\infty}$*, there is a subsequence* $\langle \varepsilon_{t_k} \rangle_{k=1}^{\infty}$*, that admits limiting strategies* $(\alpha^{i,\square})_{i=1}^{n}$

Proof. Condition 6 ensures that for all t, there is a pure-strategy equilibrium $(\alpha^{i,t})$ of the ε_t-discrete model $\mathcal{M}^{\varepsilon_t}$. Selection results (Widder (1941), Theorem 16.1) imply that for any countable $\tilde{\mathcal{X}} \times \tilde{\mathcal{S}}$ there is a subsequence $\langle \varepsilon_{t_k} \rangle_{k=1}^{\infty}$ such that $[\alpha^{i,t_k}(s_i)](x) \to [\alpha^{i,\square}(s_i)](x)$ pointwise for all i and all $(x, s_i) \in \tilde{\mathcal{X}} \times \tilde{\mathcal{S}}$. For any such sets monotonicity of $\alpha^{i,\square}$ is guaranteed by the fact that $\alpha^{i,t}$ is monotone for all i and all ε_t. The desired result follows from letting $\tilde{\mathcal{X}} = \mathcal{X}$ and $\tilde{\mathcal{S}} = \mathcal{S}$. □

Proof of Lemma 1 (main text). This is a restatement of Lemma 6, under the maintained assumption that all sequences converge. □

Lemma 7 (L^1 convergence on $\mathcal{S}$)**.** *Let* $(\alpha^{i,\square})_{i=1}^{n}$ *be limiting strategies associated with some sequence* $\langle (\alpha^{i,t})_{i=1}^{n} \rangle_{t=1}^{\infty}$ *of* $\mathcal{M}^{\varepsilon_t}$*-equilibria. Then for all* i *and all* $s_i \in \mathcal{S}$*,* $\alpha^{i,t}(s_i) \to \alpha^{i,\square}(s_i)$*.*

Proof. Suppose that $\alpha^{i,\square}(s_i)$ is continuous at some $x \in X_D$. Then for all $\lambda > 0$ there is a $\delta > 0$ such that $|[\alpha^{i,\square}(s_i)](x) - [\alpha^{i,\square}(s_i)](x + \delta')| < \lambda$ for all $\delta' \in (-\delta, \delta)$. Since $\mathcal{Q}$ is dense, there are $x_\ell, x_r \in \mathcal{X} \times (x - \delta, x + \delta)$ such that $x_\ell < x < x_r$; by pointwise convergence of $\alpha^{i,t}$ to $\alpha^{i,\square}$ on $\mathcal{X} \times \mathcal{S}$, there is a T such that $|[\alpha^{i,t}(s_i)](x') - [\alpha^{i,\square}(s_i)](x')| < \lambda$ for $x' \in \{x_\ell, x_r\}$ and all $t > T$.

The difference between $[\alpha^{i,\square}(s_i)](x_\ell)$ and $[\alpha^{i,\square}(s_i)](x_r)$ is bounded,

$$\begin{aligned} \left|\left[\alpha^{i,\square}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x_r)\right| &= \left|\left[\alpha^{i,\square}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x)\right| \\ &\quad + \left|\left[\alpha^{i,\square}(s_i)\right](x) - \left[\alpha^{i,\square}(s)\right](x_r)\right| < 2\lambda. \end{aligned}$$

This implies

$$\begin{aligned} \left|\left[\alpha^{i,t}(s_i)\right](x_\ell) - \left[\alpha^{i,t}(s_i)\right](x_r)\right| \le \Big[&\left|\left[\alpha^{i,t}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x_\ell)\right| \\ &+ \left|\left[\alpha^{i,\square}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x_r)\right| \\ &+ \left|\left[\alpha^{i,\square}(s_i)\right](x_r) - \left[\alpha^{i,t}(s_i)\right](x_r)\right| \Big] < 4\lambda. \end{aligned}$$

Since $\alpha^{i,t}(s_i)$ is monotone, this further implies that $|[\alpha^{i,t}(s_i)](x') - [\alpha^{i,t}(s_i)](x)| < 4\lambda$ for $x' \in \{x_\ell, x_r\}$. Then

$$\left|\left[\alpha^{i,t}(s_i)\right](x) - \left[\alpha^{i,\square}(s_i)\right](x)\right| \leq \Big[\left|\left[\alpha^{i,t}(s_i)\right](x) - \left[\alpha^{i,t}(s_i)\right](x_\ell)\right| \\ + \left|\left[\alpha^{i,t}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x_\ell)\right| \\ + \left|\left[\alpha^{i,\square}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x)\right|\Big] < 6\lambda.$$

Since $\lambda > 0$ may be arbitrarily small, it follows that there is T' such that $|[\alpha^{i,t}(s_i)](x) - [\alpha^{i,\square}(s_i)](x)| < \lambda$ for all $t > T'$. Then $[\alpha^{i,t}(s_i)](x) \to [\alpha^{i,\square}(s_i)](x)$ whenever $\alpha^{i,\square}(s_i)$ is continuous at x.

Since $\alpha^{i,\square}(s_i)$ is a monotone bounded function, it has at most a measure-zero set of discontinuities; hence $[\alpha^{i,t}(s_i)](x) \to [\alpha^{i,\square}(s)](x)$ for almost all x, and thus $\alpha^{i,t}(s_i) \to \alpha^{i,\square}(s_i)$. □

Limiting strategies are defined almost nowhere. However, since limiting strategies are monotonic in all dimensions and map into a compact space, they can be used to naturally define functions on all of $S \times X_D$.

Recall that a strategy $\overline{\alpha}^i$ is a supremum-limit strategy if $\alpha^{i,\square}$ is the pointwise limit of some sequence of equilibrium strategies $\langle \alpha^{i,t} \rangle_{t=1}^{\infty}$ and $\overline{\alpha}^i(s_i) = \sup_{s' < s_i} \overline{\alpha}^i(s')$ for all s_i. The choice of supremum in this construction relates to Condition 3, which ensures that small upward deviations are not discretely unprofitable. In what follows, I will fix a particular convergent sequence of discretized equilibria $\langle (\alpha^{i,t})_{i=1}^{n} \rangle_{t=1}^{\infty}$, an associated limiting strategy profile $(\alpha^{i,\square})_{i=1}^{n}$, and an associated supremum-limit strategy profile $(\overline{\alpha}^i)_{i=1}^{n}$.

Lemma 8 (Almost-sure convergence to supremum-limit strategies)**.** *For all i, $\alpha^{i,t}(s_i) \to \overline{\alpha}^i(s_i)$ with probability one.*

Proof. This proof is made notationally simpler by using measure-theoretic language. Since s_i is distributed as m independent uniform draws, it is without loss of generality to interchange Lebesgue measure and signal probability.

Note that any limiting strategy $\alpha^{i,\square}$ has at most a measure-zero set of discontinuities (Lavrič, 1993), so $\alpha^{i,t} \to \overline{\alpha}^i$. Let $\tilde{\alpha}^i$ be a completion of $\alpha^{i,\square}$ such that $[\tilde{\alpha}^i(s_i)](x) = [\alpha^{i,\square}(s_i)](x)$ whenever $[\alpha^{i,\square}(\cdot)](\cdot)$ is continuous at $(x; s_i)$; then $|[\alpha^{i,\square}(s_i)](x) - [\tilde{\alpha}^i(s_i)](x)| = 0$; adapting arguments from Lemma 7 implies that $\alpha^{i,t}(s_i) \to \tilde{\alpha}^i(s_i)$.

Let S_δ be the set signals s with δ-nonconvergent actions,

$$S_\delta = \left\{ s_i' : \lim_{t \nearrow \infty} \left|\left|\alpha^{i,t}\left(s_i'\right) - \tilde{\alpha}^i\left(s_i'\right)\right|\right| > \delta \right\},$$
$$\rightsquigarrow S_0 = \left\{ s_i' : \lim_{t \nearrow \infty} \left|\left|\alpha^{i,t}\left(s_i'\right) - \tilde{\alpha}^i\left(s_i'\right)\right|\right| > 0 \right\} = \bigcup_{w \in \mathbb{N}} S_{1/2^w}.$$

If $s \in S_0$ with positive probability, then there is $w \in \mathbb{N}$ such that $s \in S_{1/2^w}$ with positive probability. Let μ^k be the Lebesgue measure on $\mathbb{R}^k$. Consider the measure of all points of nonconvergence,

$$\begin{aligned}
&\mu^{m+m}\left(\left\{\left(x, s_i'\right) : \left|\left[\alpha^{i,t}\left(s_i'\right)\right](x) - \left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right| \not\to 0\right\}\right) \\
&= \int_S \mu^m\left(\left\{x : \lim_{t \nearrow \infty} \left|\left[\alpha^{i,t}\left(s_i'\right)\right](x) - \left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right| > 0\right\}\right) d\mu^m\left(s_i'\right) \\
&\geq \int_{s_i' \in S_{1/2^w}} \mu^m\left(\left\{x : \lim_{t \nearrow \infty} \left|\left[\alpha^{i,t}\left(s_i'\right)\right](x) - \left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right| > 0\right\}\right) d\mu^m\left(s_i'\right).
\end{aligned}$$

Let $\overline{x}, \underline{x} \in \mathbb{R}^m$ be upper and lower bounds, respectively, for X_R, and define $\Delta = ||\overline{x} - \underline{x}||$. Note that for any $s_i' \in S_{1/2^w}$, the boundedness of A^i is sufficient to imply that[24]

$$\mu^m\left(\left\{x : \lim_{t \nearrow \infty} \left|\left[\alpha^{i,t}\left(s_i'\right)\right](x) - \left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right| > 0\right\}\right) \geq \frac{1}{2^w \Delta}.$$

Then it follows that

$$\begin{aligned}
&\mu^{m+m}\left(\left\{\left(x, s_i'\right) : \left|\left[\alpha^{i,t}\left(s_i'\right)\right](x) - \left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right| \not\to 0\right\}\right) \\
&\geq \int_{s_i' \in S_{1/2^w}} \frac{1}{2^w \Delta} d\mu^m\left(s_i'\right) = \frac{\mu^m\left(S_{1/2^w}\right)}{2^w \Delta} > 0.
\end{aligned}$$

Then $\mu^{m+m}(\{(x, s_i') : |[\alpha^{i,t}(s_i')](x) - [\tilde{\alpha}^i(s_i')](x)| > 0\}) > 0$, contradicting the fact that $\alpha^{i,t} \to \tilde{\alpha}^i$. Since $\overline{\alpha}^i$ is a completion of $\alpha^{i,\square}$, it follows that $||\alpha^{i,t}(s_i) - \overline{\alpha}^i(s_i)|| \to 0$ for almost all s_i. □

Definition 8 (Upper $\mathcal{M}^\varepsilon$-approximation)**.** *The* upper $\mathcal{M}^\varepsilon$ approximation a^ε *of action $a \in A^i(s_i)$ is given by*

$$\mathsf{a}^\varepsilon(x) \in \operatorname*{arginf}_{\mathsf{a}' \in A^{i,\varepsilon}(s_i),\, \mathsf{a}' \geq a} \left|\left|a - \mathsf{a}'\right|\right|.$$

Since A^{i,ε_t} is finite, Condition 7 ensures that an upper $\mathcal{M}^{\varepsilon_t}$ approximation of a exists.

[24] Technically this must also include a term for the possible difference between $\alpha^{i,t}$ and the nearest element of $A^i(s_i)$. This difference is at most linear, and hence the argument does not change.

Lemma 9 (Interim utility approximation)**.** *There is a constant $C \in \mathbb{R}_+$ such that for any $a_i \in A^i(s_i)$,*

$$U^i\left(\mathsf{a}_i^{\varepsilon}, \alpha^{-i}; s_i\right) \geq U^i\left(a, \alpha^{-i}; s_i\right) - g\left(C\varepsilon; s_i\right).$$

Proof. Let $x \in X_D$. Then if $x - 3\varepsilon \in X_D$, there is $\mathsf{x} \in X_D^{\varepsilon}$ such that $x - 3\varepsilon < \mathsf{x} < x$. By construction, $\mathsf{a}_i^{\varepsilon}(\mathsf{x}) < a_i(\mathsf{x}) + \varepsilon$, and by monotonicity

$$\mathsf{a}_i^{\varepsilon}\left(x\right) \leq \mathsf{a}_i^{\varepsilon}\left(\mathsf{x}\right) < a_i\left(\mathsf{x}\right) + \varepsilon \leq a_i\left(x - 3\varepsilon\right) + \varepsilon.$$

Let $\overline{x}, \underline{x}$ be upper and lower bounds, respectively, for X_R. Since $\mathsf{a}_i^{\varepsilon}$ is monotone, $\mathsf{a}_i^{\varepsilon} \geq y$, and $X_R \geq 0$,

$$\begin{aligned}
||a_i - \mathsf{a}_i^{\varepsilon}|| &= \int_{X_D} \left|\mathsf{a}_i^{\varepsilon}\left(x\right) - a_i\left(x\right)\right| dx \\
&= \int_{X_D} \left|\mathsf{a}_i^{\varepsilon}\left(x\right)\right| dx - \int_{X_D} \left|a_i\left(x\right)\right| dx \\
&\leq \int_{X_D \backslash (X_D + 3\varepsilon)} \left|\overline{x}\right| dx + \int_{X_D \cap (X_D + 3\varepsilon)} \left|a_i\left(x - 3\varepsilon\right) + \varepsilon\right| \\
&\qquad\qquad - \int_{X_D \cap (X_D - 3\varepsilon)} \left|a_i\left(x\right)\right| dx - \int_{X_D \backslash (X_D - 3\varepsilon)} \left|\underline{x}\right| dx \\
&\leq 3\varepsilon\left|\overline{x}\right| + \int_{X_D \cap (X_D - 3\varepsilon)} \left|a_i\left(x\right) + \varepsilon\right| - \left|a_i\left(x\right)\right| dx \leq 4\varepsilon\left|\overline{x}\right|.
\end{aligned}$$

By construction, $\mathsf{a}_i^{\varepsilon} \in A^{i,\varepsilon}(s_i)$ and $a_i \in A^i(s_i)$, hence Condition 3 implies that

$$U^i\left(\mathsf{a}_i^{\varepsilon}, \alpha^{-i}; s_i\right) \geq U^i\left(a_i, \alpha^{-i}; s_i\right) - g\left(C\varepsilon; s_i\right).$$

□

Corollary 5 (Existence of utility approximation)**.** *For t sufficiently large, given any $a \in A^i(s_i)$, there is $\mathsf{a}^{\varepsilon_t} \in A^{i,\varepsilon_t}(s_i)$ such that*

$$U^i\left(\mathsf{a}^{\varepsilon_t}, \alpha^{-i}; s_i\right) \geq U^i\left(a, \alpha^{-i}; s_i\right) - g\left(C\varepsilon_t; s_i\right).$$

Lemma 10 (Almost no upward jumps at limit)**.** *For all agents i,*

$$\Pr_{s_i}\left(\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}(s_i),\alpha^{-i,t};s_i\right)\geq U^i\left(\overline{\alpha}^i(s_i),\overline{\alpha}^{-i};s_i\right)\right)=1.$$

Proof. This is an application of Lemma 8, Condition 4, and Corollary 5. Lemma 8 implies that with s_i-probability one,

$$\lim_{t\nearrow 0}\left|\left|\alpha^{i,t}(s_i)-\overline{\alpha}^i(s_i)\right|\right|=0.$$

Then it is sufficient to prove the claim of this Lemma under the assumption that agent i's action converges when her signal is s_i.

Assume that there is $\delta>0$ such that

$$\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}(s_i),\alpha^{-i};s_i\right)< U^i\left(\overline{\alpha}^i(s_i),\overline{\alpha}^{-i};s_i\right)-3\delta.$$

By Condition 4 there is $a\in A^i(s_i)$ such that

$$\lim_{t\nearrow\infty} U^i\left(a,\alpha^{-i};s_i\right)> U^i\left(\overline{\alpha}^i(s_i),\overline{\alpha}^{-i};s_i\right)-\delta>\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}(s_i),\alpha^{-i,t};s_i\right)+2\delta.$$

For any t, Lemma 9 implies that there is $\mathsf{a}^{\varepsilon_t}\in A^{i,\varepsilon_t}(s_i)$ such that

$$U^i\left(\mathsf{a}^{\varepsilon_t},\alpha^{-i,t};s_i\right)\geq U^i\left(a,\alpha^{-i,t};s_i\right)-g\left(C\varepsilon_t;s_i\right).$$

Putting these inequalities together, it follows that

$$\lim_{t\nearrow\infty} U^i\left(\mathsf{a}^{\varepsilon_t},\alpha^{-i,t};s_i\right)+g\left(C\varepsilon_t;s_i\right)>\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}(s_i),\alpha^{-i,t};s_i\right)+2\delta.$$

Then there is T such that for all $t>T$,

$$U^i\left(\mathsf{a}^{\varepsilon_t},\alpha^{-i,t};s_i\right)> U^i\left(\alpha^{i,t}(s_i),\alpha^{-i,t};s_i\right)+\delta.$$

Since $\mathsf{a}^{\varepsilon_t}$ is feasible for agent i in $\mathcal{M}^{\varepsilon_t}$, this implies that $\alpha^{i,t}(s_i)$ is not a best response. This can only happen with probability zero, or $\alpha^{i,t}$ is not a best response for agent i in $\mathcal{M}^{\varepsilon_t}$. □

Lemma 11 (Ex post uniform upper semicontinuity)**.** *Condition 3 is satisfied if and only if there is a continuous function* $\hat{g} : \mathbb{R}_+ \times (0,1)^m \to \mathbb{R}_+$, $\hat{g}(0;\cdot) = 0$, *such that for all agents* i, *all* $s_i \in (0,1)^m$, *all* $(a_j)_{j \neq i} \in Y^{n-1}$, $a_i \in A^i(s_i)$, *and all* $\overline{a}_i \in Y$ *with* $a_i \leq \overline{a}_i$,

$$u^i\left(a_i, a_{-i}; s_i\right) \leq u^i\left(\overline{a}_i, a_{-i}; s_i\right) + \hat{g}\left(||\overline{a}_i - a_i||\,; s_i\right).$$

Proof. Given a feasible action profile $(a_j)_{j \neq i}$ for agent i's opponents, let $\alpha^{-i} = (\alpha^j)_{j \neq i}$ be a strategy profile such that $\alpha^j(\cdot) = a_j$ for all $j \neq i$. Then Condition 3 implies the above variant.

Now, fix a strategy profile $\alpha^{-i} = (\alpha^j)_{j \neq i}$. Note that

$$\begin{aligned} U^i\left(\underline{a}_i, \alpha^{-i}; s_i\right) &= \mathbb{E}_{s_{-i}}\left[u^i\left(\overline{a}_i, \alpha^{-i}\left(s_{-i}\right); s_i\right)\right] \\ &\geq \mathbb{E}_{s_{-i}}\left[u^i\left(a_i, \alpha^{-i}\left(s_{-i}\right); s_i\right) + g\left(||\overline{a}_i - a_i||\,; s_i\right)\right] \\ &= U^i\left(a_i, \alpha^{-i}; s_i\right) + g\left(||\overline{a}_i - a_i||\,; s_i\right). \end{aligned}$$

This implies Condition 3. □

Lemma 12 (Convergence of monotone functions)**.** *Let* $\langle f^t \rangle_{t=1}^{\infty}$ *be a sequence of functions,* $f^t : S \times X_D \to X_R$, *such that* $f^t(s;\cdot)$ *is monotone for all* t *and* s. *Suppose that for all* $s \in S$, $f^t(s;\cdot) \to f^\star$. *Then* $\sup f^t = \inf\{f \in Y : \forall s \in S \; f \geq f^t(s;\cdot)\} \to f^\star$.

Proof. Suppose otherwise, and let $\bar{f}^t = \sup_{s \in S} f^t$ and $\bar{f}^\star = \lim_{t \nearrow \infty} \bar{f}^t$. Since each f^t is monotone, $\bar{f}^t$ and hence $\bar{f}^\star$ are monotone and continuous almost everywhere (Lavrič, 1993). Then if $||\bar{f}^\star - f^\star|| \neq 0$, there is $\delta > 0$ and an $x \in X_D$ such that $f^\star$ is continuous at x and $\bar{f}^\star(x) > f^\star(x) + 4\delta$, and there is $\varepsilon > 0$ such that $f^\star(x') < f^\star(x) + \delta$ for all $x' \in [x, x+\varepsilon)$.

Since $\bar{f}^t \to \bar{f}^\star$, for all T there is $t > T$ such that $\bar{f}^t(x) > f^\star(x) + 3\delta$, and thus there is $s \in S$ such that $f^t(s;x) > f^\star(x) + 2\delta$. By monotonicity, it follows that $f^t(s;x') > f^\star(x') + \delta$ for all $x' \in [x, x+\varepsilon)$, and $||f^t(s;\cdot) - f^\star|| > \varepsilon\delta$. Since ε and δ are independent of (sufficiently large) t, this contradicts $f^t(s;\cdot) \to f^\star$. □

Lemma 13 (Almost no downward jumps at limit)**.** *For all agents* i,

$$\Pr{}_{s_i}\left(\lim_{t \nearrow \infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) \leq U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right)\right) = 1.$$

Proof. This follows from Lemma 10, Conditions 3, 4, and 5, and Lemma 11. Let S_i be the set of signals for which agent i's utility in the limit is strictly greater than her utility at the limit,

$$S_i = \left\{ \lim_{t \nearrow \infty} U^i \left(\alpha^{i,t} (s_i), \alpha^{-i,t}; s_i \right) > U^i \left(\overline{\alpha}^i (s_i), \overline{\alpha}^{-i}; s_i \right) \right\}.$$

Suppose that $\Pr(s_i \in S_i) > 0$, and let $S_{-i} : S_i \rightrightarrows S^{n-1}$ be given by

$$S_{-i}(s_i) = \left\{ s_{-i} : \lim_{t \nearrow \infty} u^i \left(\alpha^{i,t} (s_i), \alpha^{-i,t} (s_{-i}); s_i \right) > u^i \left(\overline{\alpha}^i (s_i), \overline{\alpha}^{-i} (s_{-i}); s_i \right) \right\}.$$

The boundedness of u^i implies that for any $s_i \in S_i$, $\Pr(s_{-i} \in S_{-i}(s_i)) > 0$. Then by Condition 5 there is a $\delta > 0$, an agent j, a set S_j with $\Pr(s_j \in S_j) > 0$, for each $s_j \in S_j$ a set $S_{-j}(s_j)$ with $\Pr(s_{-j} \in S_{-j}(s_j)) > 0$, and for any $\lambda > 0$ a sequence $\langle \hat{\alpha}^{j,t} \rangle_{t=1}^{\infty}$ with $\hat{\alpha}^{j,t}(s_j) \in A^j(s_j)$, $||\hat{\alpha}^{j,t}(s_j) - \alpha^{j,\star}(s_j)|| < \lambda$ for all t sufficiently large, such that for all $s_j \in S_j$ and $s_{-j} \in S_{-j}(s_j)$,

$$\lim_{t \nearrow \infty} u^j \left(\hat{\alpha}^{j,t} (s_j), \alpha^{-j,t} (s_{-j}); s_j \right) > \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t} (s_j), \alpha^{-j,t} (s_{-j}); s_j \right) + 5\delta.$$

Let $\bar{\alpha}^{j,t}(s_j) = \hat{\alpha}^{j,t}(s_j) \vee \alpha^{j,t}(s_j)$. Then by Condition 3 and Lemma 11, for all $s_{-j} \in S_{-j}(s_j)$,

$$\begin{aligned}
&\lim_{t \nearrow \infty} u^j \left(\bar{\alpha}^{j,t} (s_j), \alpha^{-j,t} (s_{-j}); s_j \right) \\
&\quad \geq \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t} (s_j), \alpha^{-j,t} (s_{-j}); s_j \right) + 5\delta - g \left(\left|\left| \bar{\alpha}^{j,t} (s_j) - \hat{\alpha}^{j,t} (s_j) \right|\right|; s_j \right) \\
&\quad \geq \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t} (s_j), \alpha^{-j,t} (s_{-j}); s_j \right) + 5\delta - g (\lambda; s_j).
\end{aligned}$$

Then for λ sufficiently small,

$$\lim_{t \nearrow \infty} u^j \left(\bar{\alpha}^{j,t} (s_j), \alpha^{-j,t} (s_{-j}); s_j \right) > \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t} (s_j), \alpha^{-j,t} (s_{-j}); s_j \right) + 4\delta.$$

For $s_{-j} \notin S_{-j}(s_j)$,

$$\lim_{t \nearrow \infty} u^j \left(\bar{\alpha}^{j,t} (s_j), \alpha^{-j,t} (s_{-j}); s_j \right) > \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t} (s_j), \alpha^{-j,t} (s_{-j}); s_j \right) - g (\lambda; s_j).$$

By the law of iterated expectations,

$$U^j\left(\alpha^{j,t}(s_j),\alpha^{-j,t};s_j\right)=\Pr\left(s_{-j}\in S_{-j}(s_j)\right)\mathbb{E}_{s_{-j}}\left[u^j\left(\alpha^{j,t}(s_j),\alpha^{-j,t}(s_{-j});s_j\right)\middle|\,s_{-j}\in S_{-j}(s_j)\right]$$
$$+\Pr\left(s_{-j}\notin S_{-j}(s_j)\right)\mathbb{E}_{s_{-j}}\left[u^j\left(\alpha^{j,t}(s_j),\alpha^{-j,t}(s_{-j});s_j\right)\middle|\,s_{-j}\notin S_{-j}(s_{-j})\right].$$

Then

$$\lim_{t\nearrow\infty}U^j\left(\bar{\alpha}^{j,t}(s_j),\alpha^{-j,t};s_j\right)$$
$$>\lim_{t\nearrow\infty}U^j\left(\alpha^{j,t}(s_j),\alpha^{-j,t};s_j\right)+4\delta\Pr\left(s_{-j}\in S_{-j}(s_j)\right)-g(\lambda;s_j)\Pr\left(s_{-j}\notin S_{-j}(s_j)\right).$$

For λ sufficiently small,

$$\lim_{t\nearrow\infty}U^j\left(\bar{\alpha}^{j,t}(s_j),\alpha^{-j,t};s_j\right)>\lim_{t\nearrow\infty}U^j\left(\alpha^{j,t}(s_j),\alpha^{-j,t};s_j\right)+3\delta.$$

Appealing to Corollary 5, let $\langle\tilde{\alpha}^{j,t}\rangle_{t=1}^{\infty}$ be a sequence of strategies with $\tilde{\alpha}^{j,t}(s_j)\in A^{j,\varepsilon_t}(s_j)$ for all s_j such that, for t sufficiently large,

$$U^j\left(\tilde{\alpha}^{j,t}(s_j),\alpha^{-j,t};s_j\right)>U^j\left(\bar{\alpha}^{j,t}(s_j),\alpha^{-j,t};s_j\right)-\delta.$$

Then

$$\lim_{t\nearrow\infty}U^j\left(\tilde{\alpha}^{j,t}(s_j),\alpha^{-j,t};s_j\right)>\lim_{t\nearrow\infty}U^j\left(\alpha^{j,t}(s_j),\alpha^{-j,t};s_j\right)+2\delta.$$

It follows that for t sufficiently large,

$$U^j\left(\tilde{\alpha}^{j,t}(s_j),\alpha^{-j,t};s_j\right)>U^j\left(\alpha^{j,t}(s_j),\alpha^{-j,t};s_j\right)+\delta.$$

Then $\alpha^{j,t}(s_j)$ is not a best response for agent j when her type is s_j, against opponent play $\alpha^{-j,t}$ in model $\mathcal{M}^{\varepsilon_t}$. $\Pr(s_j\in S_j)>0$, contradicting α^t being a constrained Bayesian Nash equilibrium. □

Definition 9 (Convergent agent-types)**.** *Agent i with type s_i is a* convergent agent-type *if*

$$\lim_{t\nearrow\infty}\left|\left|\alpha^{i,t}\left(s_i\right)-\overline{\alpha}^i\left(s_i\right)\right|\right|=0,\ \textit{and}$$
$$\lim_{t\nearrow\infty}U^i\left(\alpha^{i,t}\left(s_i\right),\alpha^{-i,t};s_i\right)=U^i\left(\overline{\alpha}^i\left(s_i\right),\overline{\alpha}^{-i};s_i\right).$$

If either of these equalities does not hold, (i,s_i) is a nonconvergent agent-type*.*

Lemma 14 (Utility convergence)**.** *For all agents i,*

$$\Pr{}_{s_i}\left((i,s_i)\ \textit{ is a convergent agent-type}\right)=1.$$

Proof. For each agent i, Lemma 8 establishes that $\Pr_{s_i}(||\alpha^{i,t}(s_i)-\overline{\alpha}^i(s_i)||\to 0)=1$. Lemma 10 implies that

$$\Pr{}_{s_i}\left(\lim_{t\nearrow\infty}U^i\left(\alpha^{i,t}\left(s_i\right),\alpha^{-i,t};s_i\right)\leq U^i\left(\overline{\alpha}^i\left(s_i\right),\overline{\alpha}^{-i};s_i\right)\right)=1.$$

Lemma 13 establishes that

$$\Pr{}_{s_i}\left(\lim_{t\nearrow\infty}U^i\left(\alpha^{i,t}\left(s_i\right),\alpha^{-i,t};s_i\right)\geq U^i\left(\overline{\alpha}^i\left(s_i\right),\overline{\alpha}^{-i};s_i\right)\right)=1.$$

The result is then immediate. □

Lemma 15 (Best responses for convergent agent-types)**.** *For all agents i, $\overline{\alpha}^i(s_i)$ is a best response to $(\overline{\alpha}^j)_{j\neq i}$ with s_i-probability one.*

Proof. Suppose that agent i has a better response a_i when her type is s_i. Then there is $\delta>0$ with

$$U^i\left(a_i,\overline{\alpha}^{-i};s_i\right)>U^i\left(\overline{\alpha}^i\left(s_i\right),\overline{\alpha}^{-i};s_i\right)+4\delta.$$

By Condition 4 there is $\hat{a}_i\in A^i(s_i)$ such that

$$\lim_{t\nearrow\infty}U^i\left(\hat{a}_i,\alpha^{-i,t};s_i\right)>U^i\left(a_i,\overline{\alpha}^{-i};s_i\right)-\delta>\lim_{t\nearrow\infty}U^i\left(\alpha^{i,t}\left(s_i\right),\alpha^{-i,t};s_i\right)+3\delta.$$

Then there is T such that for all $t > T$,

$$U^i\left(\hat{a}_i, \alpha^{-i,t}; s_i\right) > U^i\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) + 2\delta.$$

By Lemma 9, there is $\hat{\mathsf{a}}^{\varepsilon_t} \in A^{i,\varepsilon_t}(s_i)$ such that

$$U^i\left(\hat{\mathsf{a}}^{\varepsilon_t}, \alpha^{-i,t}; s_i\right) > U^i\left(\hat{a}_i, \alpha^{-i,t}; s_i\right) - g\left(C\varepsilon_t; s_i\right).$$

Then there is $T' \geq T$ such that for all $t > T'$,

$$U^i\left(\hat{\mathsf{a}}^{\varepsilon_t}, \alpha^{-i,t}; s_i\right) > U^i\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) + \delta.$$

If this holds for a positive probability set of signals for agent i, this contradicts the construction of constrained Bayesian Nash equilibrium in $\mathcal{M}^{\varepsilon_t}$. □

Lemma 16 (Best responses for nonconvergent agent-types)**.** *Under Condition 8, $\overline{\alpha}^i(s_i)$ is a best response to $(\overline{\alpha}^j)_{j\neq i}$ for all agents i and signals $s_i \in S$.*

Proof. Suppose otherwise. Then there is $a_i \in A^i(s_i)$ and $\delta > 0$ such that

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i(s_i), \overline{\alpha}^{-i}; s_i\right) + 3\delta$$

If $a_i \in \underline{A}^i(s_i)$, then there is $\gamma > 0$ such that for all $s_i' < s_i$ with $||s_i - s_i'|| < \gamma$, $a_i \in A^i(s_i')$. Since utility is increasing in signal and is upper semicontinuous in action,

$$\begin{aligned} U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) &> U^i\left(\overline{\alpha}^i(s_i), \overline{\alpha}^{-i}; s_i\right) + 3\delta \\ &\geq U^i\left(\overline{\alpha}^i(s_i'), \overline{\alpha}^{-i}; s_i\right) + 3\delta \geq U^i\left(\overline{\alpha}^i(s_i'), \overline{\alpha}^{-i}; s_i'\right) + 3\delta. \end{aligned}$$

Since utility is continuous in signal, for γ sufficiently small it will be the case that whenever $||s_i - s_i'|| < \gamma$,

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i'\right) > U^i\left(\overline{\alpha}^i(s_i'), \overline{\alpha}^{-i}; s_i'\right) + 2\delta.$$

This contradicts the fact that type s_i' is almost-surely best-responding. Then $a_i \in A^i(s_i) \setminus \underline{A}^i(s_i)$.

By Condition 8 there is $a_i' \in \underline{A}^i(s_i)$ such that

$$U^i\left(a_i', \overline{\alpha}^{-i}; s_i\right) > U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) - \delta.$$

Then there is $a_i' \in \underline{A}^i(s_i)$ such that

$$U^i\left(a_i', \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + \delta.$$

The rest of the proof proceeds identically to the above, implying $a_i' \in A^i(s_i) \backslash \underline{A}^i(s_i)$, a contradiction.

□

Lemma 17 (Implied type independence)**.** *The antecedents of Theorem 3 imply Condition 8.*

Proof. Let $(\alpha^j)_{j \neq i}$ be a strategy profile for agent i's opponents, and let $a_i \in A^i(s_i) \setminus \underline{A}^i(s_i)$; by definition, $a_i \in Y$. Since u^i is continuous in signal, there is $s_i' < s_i$ such that

$$U^i\left(a_i, \alpha^{-i}; s_i'\right) > U^i\left(a_i, \alpha^{-i}; s_i\right) - \frac{1}{2}\lambda.$$

Taking as given the antecedents of Theorem 3, there is $a_i' \in A^i(s_i')$ such that

$$U^i\left(a_i', \alpha^{-i}; s_i'\right) > U^i\left(a_i, \alpha^{-i}; s_i'\right) - \frac{1}{2}\lambda.$$

Since utility is increasing in signal, it follows that

$$U^i\left(a_i', \alpha^{-i}; s_i\right) > U^i\left(a_i', \alpha^{-i}; s_i'\right) > U^i\left(a_i, \alpha^{-i}; s_i\right) - \lambda.$$

By construction, $a_i' \in \underline{A}^i(s_i)$, and Condition 8 is satisfied. □

Proof of Theorem 3 (main text). Suppose otherwise. Then there is an agent i, a signal $s_i \in S$, an action $y \in Y$, and a $\delta > 0$ such that

$$U^i\left(y, \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + 2\delta.$$

By assumption there is $a_i \in A^i(s_i)$ such that

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > U^i\left(y, \overline{\alpha}^{-i}; s_i\right) - \delta.$$

It follows that

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + \delta.$$

This directly implies that $\overline{\alpha}^i(s_i)$ is not a best response for agent i in the constrained-action game $\mathcal{M}$ when her type is s_i. Lemma 17 demonstrates that Condition 8 is satisfied whenever the antecedents of Theorem 3 hold, and Lemma 16 then implies that $\overline{\alpha}^i(s_i)$ is a best response in $\mathcal{M}$ for agent i when her signal is s_i, a contradiction. □

B Divisible-good auctions

B.1 Standard transfer rules

Lemma 18 (Standard transfer rules)**.** *Each of the example transfer rules given in Section 4 is a standard transfer rule.*

Proof. It is straightforward to see that these transfer rules are symmetric and uniformly continuous in their arguments, and that their derivatives are appropriately bounded. The random payment auction has a submodular payment rule if the discriminatory and uniform-price auctions have submodular payment rules; each of these latter two auctions has a submodular payment rule if the quantile-hybird auction has a submodular payment rule.[25]

Let b, b' be bid functions, and let $b^{\vee} = b \vee b'$ and $b^{\wedge} = b \wedge b'$. To prove modularity of expected transfer, it suffices to show that for each z,

$$\tau\left(q^{\vee}; b^{\vee}, p^{\vee}, b^{-i}, z\right) + t\left(q^{\wedge}; b^{\wedge}, p^{\wedge}, b_{-i}, z\right) \leq \tau\left(q; b, p, b_{-i}, z\right) + t\left(q'; b, p, b^{-i}, z\right),$$

where $q^{\vee} = q_i(b^{\vee}, b_{-i}; z)$, $p^{\vee} = p(b^{\vee}, b_{-i}; z)$, and similarly for the other decorated parameters. Under random priority tiebreaking, $\{q^{\vee}, q^{\wedge}\} = \{q, q'\}$ and $\{p^{\vee}, p^{\wedge}\} = \{p, p'\}$. Fixing $\alpha \in [0, 1]$, it

[25] To see this, let $\alpha = 1$ or $\alpha = 0$ to generate the discriminatory and uniform-price auctions, respectively, from a quantile-hybrid auction. In addition, McAdams (2003) establishes modularity of the discriminatory and uniform-price payment rules.

also holds that $\{p^{\alpha\vee}, p^{\alpha\wedge}\} = \{p^\alpha, p^{\alpha\prime}\}$, where p^α is the α-quantile bid when bidder i's bid is b. It is without loss to assume below that $\{q^{\alpha\vee}, q^{\alpha\wedge}\} = \{q^\alpha, q^{\alpha\prime}\}$.[26]

Without loss of generality assume that $p^\vee = p$; then $q^\vee = q$, $p^\wedge = p'$, and $q^\wedge = q'$. For simplicity of notation, suppress function arguments. Then establishing submodularity requires showing

$$p^{\alpha\vee}q^{\alpha\vee} + \int_{q^{\alpha\vee}}^{q^\vee} b^\vee dx + p^{\alpha\wedge}q^{\alpha\wedge} + \int_{q^{\alpha\wedge}}^{q^\wedge} b^\wedge dx \leq p^\alpha q^\alpha + \int_{q^\alpha}^{q} bdx + p^{\alpha\prime}q^{\alpha\prime} + \int_{q^{\alpha\prime}}^{q'} b'dx. \tag{1}$$

If $q^\vee = q$, then $b \geq b'$ for all $x \in (q', q)$. Then (1) can be rewritten as

$$p^{\alpha\vee}q^{\alpha\vee} + p^{\alpha\wedge}q^{\alpha\wedge} + \int_{q^{\alpha\vee}}^{q'} b^\vee - bdx \leq p^\alpha q^\alpha + p^{\alpha\prime}q^{\alpha\prime} + \int_{q^{\alpha\wedge}}^{q'} b' - b^\wedge dx - \int_{q^{\alpha\vee}}^{q^\alpha} bdx - \int_{q^{\alpha\wedge}}^{q^{\alpha\prime}} b'dx.$$

The simplifying assumptions on α-quantile quantities imply $p^{\alpha\vee}q^{\alpha\vee} + p^{\alpha\wedge}q^{\alpha\wedge} = p^\alpha q^\alpha + p^{\alpha\prime}q^{\alpha\prime}$. Then the above is equivalent to

$$\int_{q^{\alpha\vee}}^{q'} b^\vee - bdx \leq \int_{q^{\alpha\wedge}}^{q'} b' - b^\wedge dx - \int_{q^{\alpha\vee}}^{q^\alpha} bdx - \int_{q^{\alpha\wedge}}^{q^{\alpha\prime}} b'dx. \tag{2}$$

Note that $q^{\alpha\wedge} \leq q^{\alpha\vee}$. If $q^{\alpha\vee} = q^\alpha$, inequality (2) is

$$\int_{q^{\alpha\vee}}^{q'} b^\vee - bdx \leq \int_{q^{\alpha\wedge}}^{q'} b^\vee - bdx = \int_{q^{\alpha\wedge}}^{q'} b' - b^\wedge dx.$$

If instead $q^{\alpha\vee} = q^{\alpha\prime}$, inequality (2) is

$$\int_{q^{\alpha\vee}}^{q'} b^\vee - b'dx \leq \int_{q^{\alpha\wedge}}^{q'} b^\vee - b'dx = \int_{q^{\alpha\wedge}}^{q'} b - b^\wedge dx.$$

Then in either case, transfers are submodular. □

B.2 Proofs for equilibrium existence in divisible-good auctions

The proof of Lemma 4 (in the main text) establishes that the market clearing price is uniformly continuous in bidder i's own bid, and can be easily adapted to show that the market clearing price is uniformly continuous in all submitted bids. In light of this result, for compactness the proofs

[26] The only effect of this assumption is whether the α-quantile quantity is allocated at the α-quantile bid, or at the submitted bid. Since these are equal it is without loss to make the simplifying assumption.

below generally omit price as an argument to the transfer function τ. When a sequence of bids is converging, so too is the market price, and uniform continuity of standard transfer rules implies that the effect on price is irrelevant to convergence. All proofs can be adapted to explicitly include the effect of a change in price. Henceforth, let $\tau(q;b_i,b_{-i},z)=\tau(q;b_i,p^\star(b_i,b_{-i};z),b_{-i},z)$.

Lemma 19 (Satisfaction of Condition 3)**.** *The divisible-good auction with Lipschitz bids and a standard transfer rule satisfies Condition 3.*

Proof. For simplicitly we prove satisfaction of an ex post formulation of Condition 3 for any bid profile, implying the interim formulation given in Condition 3 (see Lemma 11). Let $b_i \in A^i(s_i)$, $b_{-i}=(b_j)_{j\neq i}$ be actions for agent i's opponents, and $\overline{b}_i \in Y$, $\overline{b}_i \geq b_i$. Define $\lambda = ||\overline{b}_i - b_i||$, and for compactness let q_i and $\overline{q}_i$ be allocations under bids b_i and $\overline{b}_i$, respectively, and $\tau(\cdot;b_i)$ and $\tau(\cdot;\overline{b}_i)$ be the associated transfers. Then the difference in ex post utility is given by

$$u^i\left(\overline{b}_i,b_{-i};s_i\right)-u^i\left(b_i,b_{-i};s_i\right)=\mathbb{E}_z\left[\int_0^{\overline{q}_i} v^i\left(x;s_i\right)dx-\tau\left(\overline{q}_i;\overline{b}_i\right)-\mathbb{E}_z\left[\int_0^{q_i} v^i\left(x;s_i\right)dx-\tau\left(q_i;b_i\right)\right]\right].$$

Allocations are monotone in bid, so $q^i(\overline{b}_i,b_{-i};z)\geq q^i(b_i,b_{-i};z)$. Since the transfer rule is standard,

$$\begin{aligned}
&u^i\left(\overline{b}_i,b_{-i};s_i\right)-u^i\left(b_i,b_{-i};s_i\right)\\
&\geq \mathbb{E}_z\left[\int_0^{q_i} v^i\left(x;s_i\right)dx-\tau\left(q_i;\overline{b}_i\right)\right]-\mathbb{E}_z\left[\int_0^{q_i} v^i\left(x;s_i\right)dx-\tau\left(q_i;b_i\right)\right]+\mathbb{E}_z\left[\int_{q_i}^{\overline{q}_i} v^i\left(x;s_i\right)-\overline{b}_i\left(x\right)dx\right]\\
&=\mathbb{E}_z\left[t\left(q_i;b_i\right)-\tau\left(q_i;\overline{b}_i\right)\right]+\mathbb{E}_z\left[\int_{q_i}^{\overline{q}_i} v^i\left(x;s_i\right)-\overline{b}_i\left(x\right)dx\right].
\end{aligned}$$

Lemma 4 implies that the left-hand term is uniformly bounded in λ. Further, since $b_i \leq v^i(\cdot;s_i)$ and $||b_i-\overline{y}_i||=\lambda$, $\int_{q_i}^{\overline{q}_i} v^i(x;s_i)-\overline{b}_i(x)dx \geq -\lambda$. This completes the proof. □

In the following let $I:\mathbb{R}^2 \rightrightarrows \mathbb{R}$ be given by $I(a,b)=(\min\{a,b\},\max\{a,b\})$, the open interval between a and b, accounting as necessary for the cases in which $a\leq b$ and $b\leq a$.

Lemma 20 (Discontinuous allocations)**.** *Let $\langle(b_{i,t})_{i=1}^n\rangle_{t=1}^\infty$ be a sequence of bid functions converging to $(b_{i,\star})_{i=1}^n$. Suppose that $\lim_{t\nearrow\infty} q^i(b_{i,t},b_{-i,t};z)\neq q^i(b_{i,\star},b_{-i,\star};z)$, and that the limit exists. Then*

$$b_{i,\star}\left(q'\right)=b_{i,\star}\left(q''\right),\ \forall q',q''\in I\left(\lim_{t\nearrow\infty} q^i\left(b_{i,t},b_{-i,t};z\right),q^i\left(b_{i,\star},b_{-i,\star};z\right)\right).$$

Proof. For any agent j, define quantities

$$q_{j,t} = q^j\left(b_j, b_{-j,t}; z\right),\ q_{j,\star} = q^j\left(b_j, b_{-j,\star}; z\right),\ \overline{q}_j = \lim_{t\nearrow\infty} q_{j,t}.$$

Assume without loss of generality that $\overline{q}_i < q_{i,\star}$; by market clearing there a nonempty set of agents J such that for all $j \in J$, $\overline{q}_j > q_{j,\star}$. Let $\delta > 0$ be such that $\lim_{t\nearrow\infty} |q_{k,t} - q_{k,\star}| > 2\delta$ for all $k \in J \cup \{i\}$.

By market clearing, it must be that for all t sufficiently large and all $j \in J$,

$$b_{j,t}\left(q_{j,\star} + \delta\right) \geq b_{i,t}\left(q_{i,\star} - \delta\right),\ \text{and}\ b_{j,t}\left(\overline{q}_j - \delta\right) \geq b_{i,t}\left(\overline{q}_i + \delta\right).$$

In the limit, it must be that

$$\lim_{t\nearrow\infty} b_{j,t}\left(q_{j,\star} + \delta\right) \geq b_{j,\star}\left(\overline{q}_j - \delta\right),\ \text{and}\ \lim_{t\nearrow\infty} b_{i,t}\left(q_{i,\star} - \delta\right) \leq b_{i,\star}\left(\overline{q}_i + \delta\right).$$

It follows that

$$\lim_{t\nearrow\infty} b_{j,t}\left(q_{j,\star} + \delta\right) \geq \lim_{t\nearrow\infty} b_{j,t}\left(\overline{q}_j - \delta\right) \geq \lim_{t\nearrow\infty} b_{i,t}\left(\overline{q}_i + \delta\right) \geq \lim_{t\nearrow\infty} b_{i,t}\left(q_{i,\star} - \delta\right). \tag{3}$$

Further, it must be the case that $\lim_{t\nearrow\infty} b_{j,t}(q_{j,\star}+\delta) \leq \lim_{t\nearrow\infty} b_{i,t}(q_{i,\star}-\delta)$. Otherwise, monotonicity and convergence together imply that

$$b_{j,\star}\left(q_{j,\star} + \delta'\right) > b_{i,\star}\left(q_{i,\star} - \delta'\right)\ \forall \delta' \in (0, \delta).$$

This contradicts the definition of $q_{j,\star}$ and $q_{i,\star}$. Then the inequalities in (3) hold with equality, and

$$\lim_{t\nearrow\infty} b_{i,t}\left(q_{i,\star} - \delta\right) = b_{i,\star}\left(\overline{q}_i + \delta\right).$$

Since this is true for all $\delta' \in (0, \delta)$, it follows that

$$\lim_{t\nearrow\infty} b_{i,t}\left(\lim_{q'\nearrow q_{i,\star}} q'\right) = b_{i,\star}\left(\lim_{q'\searrow \overline{q}_i} q'\right).$$

Then monotonicity and convergence imply that

$$b_{i,\star}\left(q'\right)=b_{i,\star}\left(q''\right),\ \forall q',q''\in I\left(\lim_{t\nearrow\infty}q^{i}\left(b_{i,t},b_{-i,t};z\right),q^{i}\left(b_{i,\star},b_{-i,\star};z\right)\right).$$

□

In what follows let u_z^i denote realized utility,

$$u_z^i\left(\tilde{b}_i,\tilde{b}_{-i};s_i,z\right)=\int_0^{q^i\left(\tilde{b}_i,\tilde{b}_{-i};z\right)}v^i\left(x;s_i\right)dx-\tau\left(q^i\left(\tilde{b}_i,\tilde{b}_{-i};z\right);\tilde{b}_i,\tilde{b}_{-i},z\right),$$
$$u^i\left(\tilde{b}_i,\tilde{b}_{-i};s_i\right)=\mathbb{E}_z\left[u_z^i\left(\tilde{b}_i,\tilde{b}_{-i};s_i,z\right)\right].$$

Showing that an inequality holds for u_z^i for all z is sufficient to show that it holds for u^i, and in turn if this holds for all $\tilde{b}_{-i}$ the inequality will hold for interim utility U^i.

Lemma 21 (Utility dominance in limit)**.** *Let $\langle(b_{j,t})_{j\neq i}\rangle_{t=1}^{\infty}$ be bid functions for agent i's opponents, converging to $b_{-i,\star}=(b_{j,\star})_{j\neq i}$. Then there is a continuous function $h:\mathbb{R}_+\to\mathbb{R}_+$, $h(0)=0$, such that for any $y\in Y$ and all $\lambda>0$,*

$$\lim_{t\nearrow\infty}u^i\left([y+\lambda]\wedge v^i\left(\cdot;s_i\right),b_{-i,t};s_i\right)\geq u^i\left(y,b_{-i,\star};s_i\right)-h\left(\lambda\right).$$

Proof. To establish this result it is sufficient to prove the above inequality with respect to u_z^i for any z. Let $\overline{y}^\lambda=[y+\lambda]\wedge v^i(\cdot;s_i)$. Note that

$$\begin{aligned}
&\lim_{t\nearrow\infty}u_z^i\left(\overline{y}^\lambda,b_{-i,t};s_i,z\right)\\
&=\lim_{t\nearrow\infty}\int_0^{q^i\left(\overline{y}^\lambda,b_{-i,t};z\right)}v^i\left(x;s_i\right)dx-\tau\left(q^i\left(\overline{y}^\lambda,b_{-i,t};z\right);\overline{y}^\lambda,b_{-i,t},z\right)\\
&=\lim_{t\nearrow\infty}\int_0^{q^i\left(y,b_{-i,\star};z\right)}v^i\left(x;s_i\right)dx-\tau\left(q^i\left(y,b_{-i,\star};z\right);y,b_{-i,\star},z\right)\\
&\qquad+\int_{q^i\left(y,b_{-i,\star};z\right)}^{q^i\left(\overline{y}^\lambda,b_{-i,t};z\right)}v^i\left(x;s_i\right)dx-\left[\tau\left(q^i\left(\overline{y}^\lambda,b_{-i,t};z\right);\overline{y}^\lambda,b_{-i,t},z\right)-\tau\left(q^i\left(y,b_{-i,\star};z\right);y,b_{-i,\star},z\right)\right].
\end{aligned}$$

Let $q_{i,t}^\lambda=q^i(\overline{y}^\lambda,b_{-i,t};z)$ and $q_{i,\star}=q^i(y,b_{-i,\star};z)$. It will suffice to show that there is a continuous

$h : \mathbb{R}_+ \to \mathbb{R}_+$, $h(0) = 0$, such that

$$\lim_{t \nearrow \infty} \left[\tau \left(q^\lambda_{i,t}; \overline{y}^\lambda, b_{-i,t}, z \right) - \tau \left(q_{i,\star}; y, b_{-i,\star}, z \right) \right] - \int_{q_{i,\star}}^{q^\lambda_{i,t}} v^i \left(x; s_i \right) dx \leq h \left(\lambda \right). \tag{4}$$

Since $b_{-i,t} \to b_{-i,\star}$, uniform continuity of transfers in opponent demand implies that for any q, $t(q; \overline{y}^\lambda, b_{-i,t}; z) \to t(q; \overline{y}^\lambda, b_{-i,\star}; z)$. Furthermore, $||\overline{y}^\lambda - \left[(y \wedge v^i(\cdot; s_i)) + \lambda \right] || \leq \overline{Q}\lambda$. Since $y \wedge v^i(\cdot; s_i) \leq y$, monotonicity and uniform continuity of transfers in own bid imply that there is a continuous $h^b : \mathbb{R}_+ \to \mathbb{R}_+$, $h^b(0) = 0$, such that for any q,

$$\lim_{t \nearrow \infty} \tau \left(q; \overline{y}^\lambda, b_{-i,t}, z \right) \leq \tau \left(q; y, b_{-i,\star}, z \right) + h^b \left(\lambda \right). \tag{5}$$

Inequality (5) transforms the left-hand side of (4) into

$$\begin{aligned} &\lim_{t \nearrow \infty} \left[\tau \left(q^\lambda_{i,t}; \overline{y}^\lambda, b_{-i,t}, z \right) - \tau \left(q_{i,\star}; y, b_{-i,\star}, z \right) \right] - \int_{q_{i,\star}}^{q^\lambda_{i,t}} v^i \left(x; s_i \right) dx \\ &\leq h^b \left(\lambda \right) + \lim_{t \nearrow \infty} \tau \left(q^\lambda_{i,t}; \overline{y}^\lambda, b_{-i,t}, z \right) - \tau \left(q_{i,\star}; \overline{y}^\lambda, b_{-i,t}, z \right) - \int_{q_{i,\star}}^{q^\lambda_{i,t}} v^i \left(x; s_i \right) dx \\ &\leq h^b \left(\lambda \right) - \lim_{t \nearrow \infty} \int_{q_{i,\star}}^{q^\lambda_{i,t}} v^i \left(x; s_i \right) - \overline{y}^\lambda \left(x \right) dx. \end{aligned}$$

Since $\lim_{\lambda' \searrow 0} h^b(\lambda') = 0$, all that remains to establish the existence of the desired h is to show

$$\lim_{t \nearrow \infty} \int_{q_{i,\star}}^{q^\lambda_{i,t}} v^i \left(x; s_i \right) - \overline{y}^\lambda \left(x \right) dx \geq 0. \tag{6}$$

Since $\overline{y}^\lambda \leq v^i(\cdot; s_i)$ by construction, if $q_{i,\star} \leq \lim_{t \nearrow \infty} q^\lambda_{i,t}$ inequality (6) is trivially satisfied. Therefore assume that $\overline{q}^\lambda_i \equiv \lim_{t \nearrow \infty} q^\lambda_{i,t} < q_{i,\star}$. Recall that $b_{-i,t} \to b_{-i,\star}$. Then for bidder i's quantity to be higher under y against $b_{-i,\star}$ than in the limit under $\overline{y}^\lambda$ against $b_{-i,t}$, it must be that $\overline{y}^\lambda(q) \leq y(q)$ for all $q \in (\overline{q}^\lambda_i, q_{i,\star})$. By construction this is only possible when $y \geq v^i(q; s_i)$ for all such q, and hence $\overline{y}^\lambda(q) = v^i(q; s_i)$ for all such q. Then $\lim_{t \nearrow \infty} \int_{q_{i,\star}}^{q^\lambda_{i,t}} v^i(x; s_i) - \overline{y}^\lambda(x) dx = 0$. In either case inequality (4) is satisfied. □

Lemma 22 (Limit surplus splitting)**.** *The divisible-good auction model satisfies Condition 5.*

Proof. Suppose that there is a sequence of strategies $\langle (\beta^{k,t})_{k=1}^n \rangle_{t=1}^\infty$ converging to the feasible strategy profile $(\beta^{\star,k})_{k=1}^{n}$ such that there is an agent i with

$$\Pr{}_{s_i}\left(\lim_{t\nearrow\infty} U^i\left(\beta^{i,t}\left(s_i\right),\beta^{-i,t};s_i\right) > U^i\left(\beta^{i,\star}\left(s_i\right),\beta^{-i,\star};s_i\right)\right) > 0.$$

Then there is a set S_i, $\Pr(s_i \in S_i) > 0$, and for each $s_i \in S_i$ a set $S_{-i}(s_i)$, $\Pr(s_{-i} \in S_{-i}(s_i)) > 0$, such that for all $s_i \in S_i$ and $s_{-i} \in S_{-i}(s_i)$,

$$\lim_{t\nearrow\infty} u^i\left(\beta^{i,t}\left(s_i\right),\beta^{-i,t}\left(s_{-i}\right);s_i\right) > u^i\left(\beta^{i,\star}\left(s_i\right),\beta^{-i,\star}\left(s_{-i}\right);s_i\right).$$

Then for these same $s = (s_i, s_{-i})$,

$$\lim_{t\nearrow\infty} \mathbb{E}_z\left[q^i\left(\beta^t\left(s\right);z\right)\right] > \mathbb{E}_z\left[q^i\left(\beta^\star;z\right)\right].$$

It follows that

$$\Pr\left(\lim_{t\nearrow\infty} q^i\left(\beta^t\left(s\right);z\right) > q^i\left(\beta^\star\left(s\right);z\right)\right) > 0.$$

Lemma 20 establishes that for any $s_i \in S_i$, $\beta^{i,\star}(s_i)$ is constant on intervals on which quantity does not converge, hence $\overline{\varphi}^{i,\star}(\cdot;s_i)$ is discontinuous at this bid level. Since $\overline{\varphi}^{i,\star}(\cdot;s_i)$ is a monotone function on a compact domain, it has at most countably-many discontinuities, so at least one such quantity interval is realized with positive probability (otherwise bidder i's utility is almost surely converging). Considering such positive-probability intervals, there is a subset of signals $\hat{S}_i \subseteq S_i$ such that these positive-probability intervals intersect, and it is without loss of generality to assume that this subset has positive measure; otherwise, the interval $[0,\bar{Q}]$ can be covered by uncountably-many disjoint sets of positive measure, a contradiction. Lastly, market clearing implies that agent i's quantity loss is some other agent's quantity gain, and since there are only a finite number of agents it is again without loss of generality to assume that in all cases at least some of the discrete gain goes to agent $j \neq i$. Then let $\hat{S}_i \subseteq S_i$ be a positive-probability set such that there are $q_{i,\ell}, q_{i,r} \in [0,\bar{Q}]$ such that for all $s_i \in \hat{S}_i$,

$$\Pr{}_{s_{-i},z}\Bigg(q^i\left(\beta^{i,\star}\left(s_i\right),\beta^{-i,\star}\left(s_{-i}\right);z\right) \leq q_{i,\ell} < q_{i,r} \\ \leq \lim_{t\nearrow\infty} q^i\left(\beta^{i,t}\left(s_i\right),\beta^{-i,t}\left(s_{-i}\right);z\right)\Bigg) > 0.$$

For $s_i \in \hat{S}_i$, let $\hat{S}_{-i}(s_i)$ be given by

$$\hat{S}_{-i}(s_i) = \Big\{ (s_{-i}, z) : \; q^i \left(\beta^{i,\star}(s_i), \beta^{-i,\star}(s_{-i}); z\right) \leq \; q_{i,\ell} < q_{i,r} \leq \lim_{t \nearrow \infty} q^i \left(\beta^{i,t}(s_i), \beta^{-i,t}(s_{-i}); z\right) \Big\}.$$

Lemma 20 implies that $\beta^{i,\star}(s_i)$ is constant on $(q_{i,\ell}, q_{i,r})$ for all $s_i \in \hat{S}_i$. Additionally, this bid must equal to the bid placed by any agent who, at the limit, receives agent i's sacrificed quantity. Then if $s_i, s_i' \in \hat{S}_i$ are such that $\beta^{i,\star}(s_i) \neq \beta^{i,\star}(s_i')$ on $(q_{i,\ell}, q_{i,r})$, it must be that $\hat{S}_{-i}(s_i) \cap \hat{S}_{-i}(s_i') = \emptyset$. From this and the fact that $\Pr(s_{-i} \in \hat{S}_{-i}(s_i)) > 0$, it follows that there is a bid level p and a positive-probability set $\tilde{S}_i \subseteq \hat{S}_i$ of agent i's signal realizations such that for all $s_i, s_i' \in \tilde{S}_i$ and $q, q' \in (q_{i,\ell}, q_{i,r})$,

$$\left[\beta^{i,\star}(s_i)\right](q) = p = \left[\beta^{i,\star}(s_i')\right](q').$$

Let S be defined as

$$S = \Big\{ (s, z) : \; q^i \left(\beta^{i,\star}(s_i), \beta^{-i,\star}(s_{-i}); z\right) \leq \; q_{i,\ell} < q_{i,r} \leq \lim_{t \nearrow \infty} q^i \left(\beta^{i,t}(s_i), \beta^{-i,t}(s_{-i}); z\right),$$
$$\text{and } \left[\beta^{i,\star}(s_i)\right](q) = \left[\beta^{i,\star}(s_i)\right](q') \;\; \forall q, q' \in (q_{i,\ell}, q_{i,r}),$$
$$\text{and } \lim_{t \nearrow \infty} q^j \left(\beta^{j,t}(s_j), \beta^{-j,t}(s_{-j}); z\right) < q^j \left(\beta^{j,\star}(s_j), \beta^{-j,\star}(s_{-j}); z\right) \Big\}.$$

Then there is an agent j, quantities $q_{j,\ell}, q_{j,r} \in [0, \bar{Q}]$ with $q_{j,\ell} < q_{j,r}$, and a set $\hat{S} \subseteq S$ with $\Pr((s,z) \in \hat{S}) > 0$ such that for all $(s, z) \in \hat{S}$,

$$\lim_{t \nearrow \infty} q^j \left(\beta^{j,t}(s_j), \beta^{-j,t}(s_{-j}); z\right) \leq q_{j,\ell} < q_{j,r} \leq q^j \left(\beta^{j,\star}(s_j), \beta^{-j,\star}(s_{-j}); z\right)$$
$$\text{and } \left[\beta^{j,\star}(s_j)\right](q) = p = \left[\beta^{j,\star}(s_j')\right](q') \;\; \forall q, q' \in (q_{j,\ell}, q_{j,r}).$$

Fix s_j and define an alternative bid $\bar{b}_\lambda^{j,t}$ by

$$\bar{b}_{j,t}^{\lambda} = \left[\left[\beta^{j,\star}(s_j) \vee \beta^{j,t}(s_j)\right] + \lambda\right] \wedge v^j(\cdot; s_j).$$

Define $d_t = ||\beta^{j,t}(s_j) - \beta^{j,\star}(s_j)||$. Since the divisible-good model with standard transfers satisfies

Condition 3, for all opponent signal realizations s_{-j} and random realizations z,

$$u_z^j\left(\bar{b}_{j,t}^{\lambda},\beta^{-j,t}\left(s_{-j}\right);s_j,z\right)\geq u_z^j\left(\beta^{j,t}\left(s_j\right),\beta^{-j,t}\left(s_{-j}\right);s_j,z\right)-g\left(d_t+\lambda\bar{Q};s_j\right).$$

Since the right-hand residual can be made arbitrarily small by letting λ be small and t be large, it will suffice to show that with positive probability the above inequality is strict, even without the residual term.

Let $(s_j,s_{-j},z)\in\hat{S}$. Since $v^j(q;\cdot)$ is strictly increasing for all q, it is without loss to assume that $[\beta^{j,\star}(s_j)](q_{j,r})<v^j(q_{j,r};s_j)$. Since standard transfers are bounded above by bids, it follows that there is $\delta>0$ such that

$$\lim_{t\nearrow\infty}u_z^j\left(\beta^{j,t}\left(s_j\right),\beta^{-j,t}\left(s_{-j}\right);s_j,z\right)<u_z^j\left(\beta^{j,\star}\left(s_j\right),\beta^{-j,t}\left(s_{-j}\right);s_j,z\right)-2\delta. \qquad (7)$$

Furthermore, for λ sufficiently small and t sufficiently large, $[\beta^{j,\star}(s_j)](q_{j,r})<\bar{b}_{j,t}^{\lambda}(q_{j,r})<v^j(q_{j,r};s_j)$, hence

$$\lim_{t\nearrow\infty}q^j\left(\bar{b}_{j,t}^{\lambda},\beta^{-j,t}\left(s_{-j}\right);z\right)\geq q^j\left(\beta^{j,\star}\left(s_j\right),\beta^{-j,\star}\left(s_{-j}\right);z\right)\equiv q_{j,\star}\geq q_{j,r}.$$

Uniform continuity of standard transfers in own and opponents' bids implies that for λ sufficiently small,

$$\lim_{t\nearrow\infty}\tau\left(q;\bar{b}_{j,t}^{\lambda},\beta^{-j,t}\left(s_{-j}\right),z\right)-\tau\left(q;\beta^{j,\star}\left(s_j\right),\beta^{-j,\star}\left(s_{-j}\right),z\right)<\delta.$$

Since $\bar{b}_{j,t}^{\lambda}\geq\beta^{j,t}(s_j)$ is bounded above by marginal value $v^j(\cdot;s_j)$,

$$\begin{aligned}
&\lim_{t\nearrow\infty}u_z^j\left(\bar{b}_{j,t}^{\lambda},\beta^{-j,t}\left(s_{-j}\right);s_j,z\right)\\
&\geq\lim_{t\nearrow\infty}u_z^j\left(\beta^{j,\star}\left(s_j\right),\beta^{-j,\star}\left(s_{-j}\right);s_j,z\right)\\
&\qquad-\left[\tau\left(q_{j,\star};\bar{b}_{j,t}^{\lambda},\beta^{-j,t}\left(s_{-j}\right),z\right)-\tau\left(q_{j,\star};\beta^{j,\star}\left(s_j\right),\beta^{-j,t}\left(s_{-j}\right),z\right)\right]\\
&\qquad+\int_{q_{j,\star}}^{q^j\left(\bar{b}_{j,t}^{\lambda},\beta^{-j,t}(s_{-j});z\right)}v^j\left(x;s_j\right)-\bar{b}_{j,t}^{\lambda}\left(x\right)dx\geq u_z^j\left(\beta^{j,\star}\left(s_j\right),\beta^{-j,\star}\left(s_{-j}\right);s_j,z\right)-\delta. \qquad (8)
\end{aligned}$$

Putting together (7) and (8) gives

$$\lim_{t\nearrow\infty} u_z^j\left(\bar{b}_{j,t}^{\lambda}, \beta^{-j,t}(s_{-j}); s_j, z\right) > \lim_{t\nearrow\infty} u_z^j\left(\beta^{j,t}(s_j), \beta^{-j,t}(s_{-j}); s_j, z\right) + \delta.$$

Since this δ improvement can be realized with positive probability while costs can be made arbitrarily small, it follows that

$$\lim_{t\nearrow\infty} u^j\left(\bar{b}_{j,t}^{\lambda}, \beta^{-j,t}(s_{-j}); s_j\right) > \lim_{t\nearrow\infty} u^j\left(\beta^{j,t}(s_j), \beta^{-j,t}(s_{-j}); s_j\right).$$

□

Lemma 23 (Divisible-good type insensitivity)**.** *Divisible-good auctions with standard transfers satisfy Condition 8.*

Proof. Let $b_i \in A^i(s_i)$, and for $\lambda > 0$ consider an alternative bid function $\underline{b}_i^{\lambda} = [v^i(\cdot; s_i) - \lambda]_+ \wedge b_i$. Then $\underline{b}_i^{\lambda} \in \underline{A}^i(s_i)$, and $\underline{b}_i^{\lambda} \leq b_i$. Consider any opponent bid profile $b_{-i} = (b_j)_{j \neq i}$. Since standard transfers are monotone in own bid, for any q and z

$$\tau\left(q; \underline{b}_i^{\lambda}, b_{-i}, z\right) \leq \tau\left(q; b_i, b_{-i}, z\right).$$

Then if $u_z^i(\underline{b}_i^{\lambda}, b_{-i}; s_i, z) < u_z^i(b_i, b_{-i}; s_i, z)$, it must be that $\underline{q}_i^{\lambda} \equiv q^i(\underline{b}_i^{\lambda}, b_{-i}; z) < q^i(b_i, b_{-i}; z) \equiv q_i$. Write

$$\begin{aligned} u_z^i(b_i, b_{-i}; s_i, z) &= u_z^i\left(\underline{b}_i^{\lambda}, b_{-i}; s_i, z\right) + \left[\tau\left(\underline{q}^{\lambda}; \underline{b}_i^{\lambda}, b_{-i}, z\right) - \tau\left(q_i; b_i, b_{-i}, z\right)\right] + \int_{\underline{q}_i^{\lambda}}^{q_i} v^i(x; s_i)\, dx \\ &\leq u_z^i\left(\underline{b}^{\lambda}, b_{-i}; s_i, z\right) + \int_{\underline{q}_i^{\lambda}}^{q_i} v^i(x; s_i) - p(b_i, b_{-i}; z)\, dx. \end{aligned} \tag{9}$$

By market clearing, it must be that $\underline{b}_i^{\lambda}(\underline{q}_i^{\lambda}) \leq p(b_i, b_{-i}; z)$. Bid montonicity and inequality (9) imply

$$u_z^i(b_i, b_{-i}; s_i, z) \leq u_z^i\left(\underline{b}_i^{\lambda}, b_{-i}; s_i, z\right) + \int_{\underline{q}_i^{\lambda}}^{q_i} v^i(x; s_i) - \underline{b}_i^{\lambda}(x)\, dx. \tag{10}$$

Furthermore, market clearing implies that for $q \in (\underline{q}_i^{\lambda}, q_i)$, $\underline{b}_i^{\lambda}(q) < b_i(q)$. Then for all such q,

$\underline{b}_i^{\lambda}(q) = v^i(q; s_i)$. Then (9) becomes

$$u_z^i(b_i, b_{-i}; s_i, z) \le u_z^i\left(\underline{b}_i^{\lambda}, b_{-i}; s_i, z\right) + \left(q_i - \underline{q}_i^{\lambda}\right)\lambda \le u_z^i\left(\underline{b}_i^{\lambda}, b_{-i}; s_i, z\right) + \overline{Q}\lambda.$$

It follows that against any opponent strategy profile $\beta^{-i} = (\beta^j)_{j \neq i}$,

$$U^i\left(\underline{b}_i^{\lambda}, \beta^{-i}; s_i\right) \ge U^i\left(b_i, \beta^{-i}; s_i\right) - \overline{Q}\lambda.$$

□

Proof of Corollary 4 (main text). That quantity is utility-relevant follows from the logic employed in the proof of Lemma 22, which establishes Condition 5. In particular, if quantity is not converging for a positive-measure set of signal realizations, it is without loss to assume that agent i loses quantity in the limit. Then for some realizations of agent signals, the lost quantity intervals overlap, and Lemma 20 then implies that for some of these type realizations the lost quantity intervals overlap at exactly the same bid level. Then because marginal values are strictly monotone in signal, a positive-measure subset of these signal realizations is such that agent i's utility is not converging, implying that quantity is utility-relevant.

Convergence of bidding strategies implies that if $\pi(y^\star(s), z) \neq \lim_{t \nearrow \infty} \pi(y^t(s); z)$, then $q(y^\star(s); z) \neq \lim_{t \nearrow \infty} q(y^t(s); z)$. Since q is utility-relevant, seller revenue π is utility-relevant. □

B.3 Discretized model

Lemma 24 (Satisfaction of Condition 7)**.** *The ε-discretized model $\mathcal{M}^\varepsilon = (n, u, X^\varepsilon, A^\varepsilon, F)$ set forth in Section 4 satisfies Condition 7.*

Proof. Closure and the lattice structure of $A^{i,\varepsilon}(s_i)$ follow immediately from its definition.

Let $b \in A^i(s_i)$, and let $b_\varepsilon : X_D^\varepsilon \to X_R^\varepsilon$ be given by

$$b_\varepsilon(q) = \left\lfloor \frac{b(q) + \varepsilon^2}{\varepsilon^2} \right\rfloor \varepsilon^2.$$

Since $b \le v^i(\cdot; s_i)$, $b_\varepsilon \le v^i(\cdot; s_i) + \varepsilon^2$ and $b_\varepsilon \in A^{i,\varepsilon}(s_i)$. Then $b_{\varepsilon_t} \searrow b$ with $b_{\varepsilon_t} \ge a$ for all t.

Lastly, let $\langle b_t \rangle_{t=1}^\infty$ be a sequence of functions, $b_t \in A^{i,\varepsilon_t}(s_i)$ for all t, and assume that $b_t \to b^\star$.

Since each b_t is monotone, $b^\star$ is monotone and $b^\star \leq v^i(\cdot; s_i)$ almost everywhere. Then $b^\star$ is L_1 equivalent to a function $\hat{b}^\star \in A^i(s_i)$, and $\langle b_t \rangle_{t=0}^{\infty}$ converges in $A^i(s_i)$. □

Lemma 25 (Satisfaction of Condition 6)**.** *The ε-discretized model $\mathcal{M}^\varepsilon$ admits a monotone pure strategy Bayesian Nash equilibrium.*

Proof. By Proposition 4.4 of Reny (2011) it is sufficient to show that bidder i's utility function is weakly quasisupermodular and satisfies weak single crossing.

Weak single crossing is straightforward. Let $b_i' \geq b_i$; then $q^i(b_i', \cdot; \cdot) \geq q^i(b_i, \cdot; \cdot)$, and

$$
\begin{aligned}
& U^i\left(b_i', \beta^{-i}; s_i\right) \geq U^i\left(b_i, \beta^{-i}; s_i\right) \\
\iff \quad & \mathbb{E}\left[\int_{q^i(b_i, b_{-i}; z)}^{q^i\left(b_i', b_{-i}; z\right)} v^i\left(x; s_i\right) dx\right] \geq \mathbb{E}\left[\tau\left(q^i\left(b_i', b_{-i}; z\right); b_i', b_{-i}, z\right) - \tau\left(q^i\left(b_i, b_{-i}; z\right); b_i, b_{-i}, z\right)\right].
\end{aligned}
$$

The left-hand side is strictly increasing in s_i while the right-hand side is constant in s_i, and weak single crossing is established.

Weak quasisupermodularity derives from the observation that, given any bids b_i and b_i' and any realization z, $\{q^i(b_i, b_{-i}; z), q^i(b_i', b_{-i}; z)\} = \{q^i(b_i \vee b_i', b_{-i}; z), q^i(b_i \wedge b_i', b_{-i}; z\}$, and the presumed submodularity of the standard transfer rule. □