Hybrid Mechanisms for the Sale of Divisible Goods

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Abstract

I provide a closed-form expression for equilibrium bids in hybrid auctions, where the payment rule is a convex combination of discriminatory and uniform price payment rules. All equilibria are symmetric, and the closed-form expression encapsulates all possible equilibria. The class of valid solutions to the expression suggests that the equilibrium multiplicity problem, observed in uniform price auctions, smoothly disappears as an auction becomes discriminatory, or as the market becomes large. When marginal values and bids are both linear, I show that discriminatory auctions strictly revenue-dominate any other hybrid mechanism, including pure uniform price auctions. Even if bids are not linear, linear marginal values imply that the discriminatory auction raises strictly more revenue than any equilibrium of the uniform price auction. In large markets all hybrid auctions have incentives that are identical to the discriminatory auction, scaled by the extent of price discrimination, and all hybrid auctions are revenue equivalent.

1 Introduction

Multi-unit auctions are commonly used to allocate homogeneous goods, including government debt, pollution rights, and electricity generation. The two most commonly implemented multi-unit auction formats are uniform price and discriminatory (Brenner et al., 2009). In each of these auctions bidders submit demand curves to the seller, and these demand curves are used to determine a market clearing price and associated quantities. In a uniform price auction each bidder pays the (constant) market-clearing price for each unit she receives, while in a discriminatory auction the bidder pays her bid for each unit she receives. Which of these mechanisms generates better outcomes remains both theoretically and empirically ambiguous. These auction formats allocate substantial financial value (in 2018, the U.S. Treasury auctioned over $10 trillion in securities), and specific guidance may have a significant impact on auction proceeds.

I investigate convex combinations of the uniform price and discriminatory auctions, in which bidders' payments are convex combinations of their discriminatory and uniform price payments. I

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obtain a closed-form expression for equilibrium bids, and prove that all equilibria are symmetric.[1] The first order conditions and closed-form expression for equilibrium bids allow theoretical exploration of the properties of hybrid auctions. First, I show that equilibrium multiplicity smoothly transitions to equilibrium uniqueness as the payment rule becomes more discriminatory, or as the market becomes large. The range of sustainable equilibrium prices is increasing in market quantity, and decreasing in price discrimination. Second, in large markets, bidding incentives are qualitatively identical across all hybrid auction formats, scaled by the weight on discriminatory transfers. In large markets, all hybrid auctions are revenue equivalent. Finally, when bids and marginal values have the same functional form, I show equilibrium revenue is monotonically increasing as payments become discriminatory, and the unique equilibrium of the discriminatory auction generates strictly greater revenue than all equilibria of the uniform price auction. I also show that the "at least three bidders" condition typically applied to uniform price auctions (see Kyle (1989), Ausubel et al. (2014), and others) is closely tied to the shape of marginal values.

Previous theoretical work has considered the hybrid auction model, but my approach is distinguished in two ways. In early analyses (Viswanathan and Wang, 2002; Wang and Zender, 2002) the hybrid auction model is used as a tool to unify the analysis of discriminatory and uniform price auctions, but the potential for strict hybridization is passed over. More recent work (Armantier and Sbaï, 2009; Ruddell et al., 2016) formally considers the possibility of strict hybridization, but does not consider the theoretical properties of hybrid mechanism equilibria.[2] By contrast, I explicitly consider the theoretical properties of strict hybridizations which are neither purely discriminatory nor purely uniform price, and derive comparative statics on this hybridization. Additionally, where the literature tends to assume that marginal values are constant for all units, I assume that bidders' marginal values are decreasing in the quantity received. Larger quantities of treasury securities, for example, serve as less effective hedges against market uncertainty, and may be marginally less valuable.

My model of divisible-good auctions is similar to those in Klemperer and Meyer (1989), Holmberg (2009), and Pycia and Woodward (2019). Bidders are symmetric and symmetrically-informed, and have strictly decreasing marginal value for quantity. There is an exogenous distribution of supply, with convex support. Bidders submit demand curves, defined on the support of supply. Aggregate supply is realized, and the auctioneer uses submitted bids to determine the market clearing price and quantities.[3] Bidders pay to the seller a convexification of their discriminatory and

[1] Unlike results in single-unit auctions (e.g., Lizzeri and Persico (2000)) equilibrium symmetry holds even in the (pure) uniform price auction, but the arguments are distinct when the discriminatory allocation receives strictly positive weight. Symmetry in the uniform price auction was first observed in a related context by Klemperer and Meyer (1989).

[2] Armantier and Sbaï (2009) conduct a counterfactual analysis of French treasury auctions, and compare outcomes in different hybrid auction formats. The hybrid auction I study is equivalent to their "α-discriminatory" auction. Ruddell et al. (2016) consider the possibility that the seller taxes apparent bidder surplus. I derive an equivalent first order condition, then formally derive equilibrium bidding strategies, and consider the general implications of hybridization.

[3] Quantity allocations are identical in the discriminatory and uniform price auction formats, so hybridization does not affect the quantity realized (conditional on submitted bids).

uniform price transfers: they pay proportion α of the area under their demand curve (the discriminatory transfer), and proportion $1-\alpha$ of their quantity at the market clearing price (the uniform price transfer). I refer to α as the extent of price discrimination.

Equilibrium bidding incentives in the hybrid auction are a convex combination of discriminatory and uniform price bidding incentives. However, because equilibrium bids are the solution to a differential equation relating elasticity to marginal utility, this does not imply that equilibrium bids are a convex combination of discriminatory and uniform price equilibrium bids. My equilibrium analysis begins by showing that all pure-strategy equilibria are in symmetric strategies. The argument for equilibrium symmetry is similar to that for discriminatory auctions (Pycia and Woodward, 2019): asymmetric bid profiles induce profitable deviations through discontinuous elasticities for small quantities.[4] Equilibrium symmetry reduces the first order conditions from a n-dimensional differential equation to a single-dimensional differential equation. I provide a closed form solution to this equation, up to an integration constant. Solutions to this equation are smoothly affected by the extent of price discrimination α.

At the extremes, the equation for equilibrium bids conforms to what is known about divisible-good auctions: the discriminatory auction admits a unique equilibrium, and uniform price equilibria are ex post. While equilibrium bids are expressable in closed form, they do not in general have a simple algebraic formulation, except in the pure discriminatory and pure uniform price cases. I make further progress in equilibrium analysis by restricting attention to the *polynomial-Lomax* model, in which marginal values are polynomials and the distribution of supply is negative Lomax. The order of the marginal value polynomials is unrestricted, and therefore very general. The negative Lomax distribution allows for arbitrary concentration of supply at low or high levels, but implies a monotone density function. In the polynomial-Lomax model, all equilibrium bids are polynomial in quantity, plus a potential homogeneous term. The polynomial coefficients are recursively defined, hence simple to compute, and the homogeneous term has a simple analytical form. The polynomial coefficients contain terms inversely proportional to the order of the marginal value polynomial less the number of bidders, implying that uniform price auction equilibria exist only when the number of bidders is at least as large as the order of the marginal value polynomial.

With this characterization in hand, I study the effect of hybridization on auction outcomes. To begin, I show that equilibrium multiplicity is smoothly affected by hybridization: the set of allowable minimum market clearing prices is shrinking in the extent of price discrimination α.[5] This result follows from an analysis of bidding incentives at large quantity realizations, where bidders' standard first order conditions are necessary but not sufficient for optimality.[6] Careful

[4]Equilibrium symmetry in the uniform price auction follows an approach essentially identical to that in Klemperer and Meyer (1989). For small quantities, nondegenerate hybrid auctions have more analytical similarity to discriminatory auctions than to uniform price auctions.

[5]Although uniqueness is a binary concept, for consistency with the literature I discuss decreased equilibrium multiplicity as increased uniqueness. Thus uniform price auction equilibria are nonunique (Klemperer and Meyer (1989), Back and Zender (1993)), the discriminatory auction equilibrium is unique (Wang and Zender (2002), Pycia and Woodward (2019)), and for any strict hybridization, equilibrium is more unique than in the uniform price auction, but still (generally) nonunique.

[6]Bidders' first order conditions for interior realizations of quantity are derived from the calculus of variations. The

study of the bidders' first-order conditions provides further results on uniqueness. First, the range of feasible equilibrium prices, as measured by the spread from high to low, is increasing in quantity realization. In markets where the distribution of supply is concentrated near its maximum (as in treasury markets with noncompetitive demand), equilibrium selection may lead to large variance in realized market prices. Second, the range of feasible equilibrium prices is decreasing in price discrimination, due to increases in both bid elasticity and equilibrium uniqueness. Thus equilibrium prices are potentially more volatile in uniform price auctions than in discriminatory auctions.

To analyze large markets, I hold the per-capita distribution of supply constant and increase the number of bidders. Two countervailing effects influence equilibrium uniqueness. On the one hand, as the number of bidders increases, the space of feasible equilibrium prices unambiguously increases. An increase in the number of bidders decreases bid elasticity, and the endpoint condition identified earlier is unaffected.[7] The set of feasible initial conditions is unchanging while the slope of bids is increasing, thus the set of feasible prices is growing. On the other hand, since the slope of bids is increasing, low prices are realized with decreasing probability, even in low-bid equilibria. Although the set of feasible prices is increasing, while the probability of certain realizations is going to zero. In the limit, with an infinite number of bidders, I prove that equilibrium is unique, for any hybridization. The low-price, low-probability effect dominates, and the only prices observable are those achievable in the seller-optimal equilibrium.[8] Incentives in large-market hybrid auctions are implementation-scaled versions of discriminatory auction incentives, and all hybrid auctions generate the same expected revenue.

Finally, I apply the linear-Lomax model to analyze the effect of hybridization on expected revenue. When bids must also be linear, I show that equilibrium expected revenue is strictly increasing in discriminatory implementation. This is not because the seller is price discriminating against fixed bids, but is net of the changes in bidding behavior induced by discriminatory auction incentives. Except in the pure discriminatory auction, linear equilibria are not seller-optimal, and I also compare expected revenues in seller-optimal, "maximum bid" equilibria of the discriminatory and uniform price auctions. As it turns out, the discriminatory auction generates greater expected revenue than even the seller-optimal equilibrium of the uniform price auction.

These results have implications for the implementation of divisible-good auctions. First, equilibrium multiplicity is smoothly affected by discriminatory implementation. This contrasts with results for single-unit auctions (Plum, 1992; Lizzeri and Persico, 2000), where even a small pro-

calculus of variations assumes a fixed initial condition for optimal solutions, but in divisible-good auctions the initial condition must be derived from the theory, and is related to the solution *given the initial condition*. This circularity implies that the standard first-order approach may be improved upon, near the endpoints of a bidder's demand curve.

[7] Holding fixed the bid function submitted by a symmetric bidders, increasing the number of bidders increases the incentives to marginally increase bids. In equilibrium this increased incentive is exactly offset by decreased bid elasticity, and the endpoint condition is unaffected by the number of bidders.

[8] Pycia and Woodward (2019) show that seller-optimal equilibria of the discriminatory and uniform price auctions are revenue-equivalent, conditional on the seller being able to set the distribution of quantity. If the seller cannot also implement a reserve price, there generally exist equilibria of the uniform price auction which are revenue-dominated by the unique equilibrium of the discriminatory auction. By contrast, my seller is passive, and cannot affect the distribution of quantity.

portion of discriminatory implementation implies equilibrium uniqueness. In the divisible-good context, a little bit does not go a long way. Second, the space of equilibrium prices is increasing in uniformity, somewhat at odds with conventional belief that uniform price prices have low variance, and are easy to predict (Friedman, 1991; Lotfi and Sarkar, 2016). Third, in large markets mechanism selection has little effect on expected revenues, at least within the space of hybrid auctions. This is in line with earlier large-market analyses (Swinkels, 2001), and provides additional meta-evidence in support of ambiguous counterfactual revenue analyses,[9] as well as recent claims that the gains from mechanism selection may be small (Hortaçsu et al., 2018). In uniform price auctions, large-market bids are essentially truthful, implying that the "truthful reporting" upper bounds on counterfactual uniform price bids (Hortaçsu and McAdams (2010), and others) may be relatively tight. Fourth, although the increase in expected revenue is small when markets are large, discriminatory auctions unambiguously raise more revenue than uniform price auctions, when demand is linear and supply is concentrated. This adds some clarity to the ambiguity surrounding revenue and efficiency rankings in divisible-good auctions (Ausubel et al., 2014). [10]

This paper proceeds by formally defining the model of divisible-good auctions with a hybrid allocation rule. In Section 3, I prove basic features of equilibrim, provide a closed form solution for equilibrium bid functions, and introduce *conjugate* bid functions. Section 4 studies equilibrium uniqueness, large markets, and the effect of hybridization on revenue in the linear-Lomax model. Most proofs are provided in the appendix.

2 Model

There are $n \geq 3$ bidders, $i \in \{1, ..., n\}$, participating in an auction for a perfectly-divisible good.[11] Bidder i has marginal value function $v^i \equiv v$, where her marginal value for quantity q is $v(q)$. v is weakly positive and monotonically decreasing in q. The available market quantity Q is stochastic and drawn according to the cumulative distribution function F, with support $[0, \overline{Q}]$ and strictly positive density f. Denote the per capita maximum supply by $Q^\mu \equiv \overline{Q}/n$. Each bidder's utility is quasilinear in the transfer she makes to the seller: if bidder i obtains quantity q_i and makes transfer t_i, her utility is

$$\tilde{u}\left(q_i, t_i\right) = \int_0^{q_i} v\left(x\right) dx - t_i. \tag{1}$$

I denote the (efficient) aggregate marginal value for quantity Q by $\hat{v}(Q) = v(Q/n)$.

[9] Empirical studies of multi-unit auctions have obtained context-dependent results. Février et al. (2002), Kang and Puller (2008), and Marszalec (2017) find discriminatory auctions raise more revenue than uniform price auctions, Armantier and Sbaï (2006), Castellanos and Oviedo (2008), and Armantier and Sbaï (2009) find the opposite, and Hortaçsu and McAdams (2010) find no statistically significant difference.

[10] Relatedly, mechanism commitment has limited power in divisible-good auctions. While a seller could solicit uniform price bids and manipulate market outcomes to improve revenue (Akbarpour and Li, 2019; Woodward, 2019), a untrustworthy seller faced with rational buyers strictly prefers to implement a discriminatory auction, and the commitment question is moot.

[11] In a (pure) uniform price auction with linear marginal values, there may exist no pure-strategy equilibria when there are fewer than three bidders. In practice these auctions generally have at least three participants. I generalize the at-least-three-bidders condition following Theorem 3.

Each agent i submits a continuous and decreasing bid function $b^i : [0, \overline{Q}] \to \mathbb{R}_+$ to the auctioneer; b^i is strictly decreasing wherever it is strictly positive.[12] Because the bid function b^i is strictly decreasing, it admits a decreasing and continuous inverse φ^i. These inverse functions implicitly define the market-clearing price $p^\star : [0, \overline{Q}] \to \mathbb{R}_+$,

$$\sum_{i=1}^{n} \varphi^i \left(p^\star \left(Q\right)\right) = Q.$$

After soliciting bid functions, random quantity Q is realized and the seller computes the market clearing price $p^\star(Q)$ and associated quantities $q_i = \varphi^i(p^\star(Q))$.[13] Each bidder is awarded her market clearing quantity, and pays the seller an α-hybridization of the payments she would make in discriminatory and uniform price auctions. In a discriminatory auction, each bidder would pay her bid for each unit she obtained, and in a uniform price auction, each bidder would pay the (constant) market clearing price for each unit she obtained. In the hybrid auction, the bidder pays proportion α of her discriminatory payment, and proportion $1 - \alpha$ of her uniform price payment,

$$t_i = \alpha \int_0^{q_i} b^i \left(x\right) dx + \left(1 - \alpha\right) p^\star q_i. \tag{2}$$

When $\alpha = 0$, the hybrid auction is a (pure) uniform price auction, and when $\alpha = 1$, the hybrid auction is a (pure) discriminatory auction.

Because bids are continuous, if bidder i submits bid b^i and obtains quantity q_i, the market clearing price is $p^\star = b^i(q_i)$. Substituting (2) into (1), the bidder's utility from obtaining quantity q at market price $p^\star = b^i(q_i)$ is

$$u\left(q_i; b^i\right) = \int_0^{q_i} v\left(x\right) dx - \left[\alpha \int_0^{q_i} b^i\left(x\right) dx + \left(1 - \alpha\right) q_i b^i\left(q_i\right)\right] dx.$$

I constrain attention to symmetric pure-strategy Nash equilibria, in which the equilibrium bid profile $(b^i)_{i=1}^n$ satisfies, for each bidder i,

$$b^i \in \underset{b}{\operatorname{argmax}}\, \mathbb{E}_{q_i}\left[u\left(q_i; b\right)\right].$$

2.1 Polynomial-Lomax model

For parametric analysis I restrict attention to the *polynomial-Lomax model.* In the polynomial-Lomax model, marginal values are piecewise polynomial in quantity, and the distribution of quantity is negative Lomax with parameter $\lambda > 0$. Given coefficients $\mathbf{v} = (v_k)_{k=0}^{\bar{k}}$, let $\bar{q}_v$ be the smallest

[12]In the appendix I show that, even if discontinuous and weakly decreasing bids are permitted, all symmetric equilibria are in continuous, strictly decreasing bids. Since Proposition 1 establishes that all equilibria are symmetric, assuming strictly decreasing bid functions is essentially without loss of generality. When bids are strictly decreasing, tiebreaking is never necessary and market prices and quantities can be identified by inverse bid functions. The assumption of strictly decreasing bids is made to simplify definitions of equilibrium outcomes.

[13]By convention, superscripts indicate functions and subscripts indicate quantities.

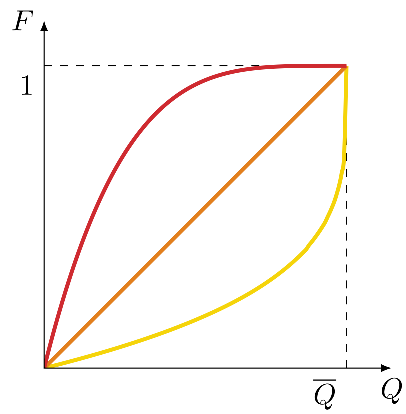

Figure 1: The cumulative distribution of the negative Lomax distribution with parameter $\lambda \in \{1/4, 1, 4\}$. The Lomax distribution is concentrated near $\overline{Q}$ when $\lambda \ll 1$, uniform when $\lambda = 1$, and concentrated near 0 when $\lambda \gg 1$.

positive root of the polynomial $\sum_{k=0}^{\bar{k}} v_k q^k$. Then marginal values and the distribution of quantity are given by

$$v(q) = \begin{cases} \sum_{k=0}^{\bar{k}} v_k q^k & \text{if } q \leq \bar{q}_v, \\ 0 & \text{otherwise;} \end{cases} \qquad F(Q) = 1 - \left(\frac{\overline{Q} - Q}{\overline{Q}}\right)^{\lambda}.$$

I place no restriction on the order of the polynomial $\bar{k}$, so polynomial marginal values are fairly general: the only constraint is that v must be decreasing in quantity. On the other hand, the Lomax distribution of supply may be restrictive: while it allows for arbitrary probability mass at either tail of the distribution (see Figure 1), its probability density function is always monotone.

3 Equilibrium

Bidder i's objective function is

$$\max_b \mathbb{E}_{q_i}\left[\int_0^{q_i} v(x) - \alpha b(x)\, dx - (1-\alpha)\, q_i b(q_i) \middle| b\right].$$

This optimization problem is simplified by integrating by parts (Février et al., 2002; Hortaçsu, 2002; Pycia and Woodward, 2019). Letting G^i be the equilibrium distribution of bidder i's allocation conditional on her bid b, $G^i(q;b) = \Pr(q_i \leq q|b)$, her objective function is

$$\max_b \int_0^{\overline{Q}} \left(v(q) - \alpha b(q) - (1-\alpha)\left(b(q) + q b'(q)\right)\right)\left(1 - G^i(q;b)\right) dq. \tag{3}$$

The incentives corresponding to the bid for quantity q follow from application of the calculus of variations to the maximization problem in (3).

Lemma 1 (Convexified incentives)**.** *Bidding incentives in the hybrid auction are the weighted sum* *of incentives in the uniform-price and discriminatory auctions,*

$$\underbrace{\left(\left(v\left(q\right)-b^{i}\left(q\right)\right)+\left(\frac{1-G^{i}\left(q;b^{i}\right)}{\left(n-1\right)G_{q}^{i}\left(q;b\right)}\right)b_{q}^{i}\left(q\right)\right)}_{\text{discriminatory incentives}}\alpha+\underbrace{\left(\left(v\left(q\right)-b^{i}\left(q\right)\right)+\left(\frac{q}{n-1}\right)b_{q}^{i}\left(q\right)\right)}_{\text{uniform price incentives}}\left(1-\alpha\right)=0.$$

Hybrid auctions not only specify transfers which are convex combinations of discriminatory and uniform price transfers, but also induce incentives which are convex combinations of discriminatory and uniform price incentives. Convex incentives do not imply that equilibrium bids are convex combinations of discriminatory and uniform price bids, and equilibrium bids have a nontrivial dependence on the degree of hybridization.

In equilibrium, each bidder is best responding to the distribution of residual supply generated by her opponents' bidding strategies, taking into account her own market power. This implies that solving for equilibrium bid profiles amounts to solving an n-dimensional differential system. This is apparent in Lemma 1, where the bidder's first order conditions depend on her own bid, through b^i and b^i_q, and on her opponents' bids, through G^i. The first step toward obtaining a closed form expression for equilibrium bids is to show that all equilibria are symmetric, reducing the differential system to a single dimension.

Proposition 1 (Equilibrium symmetry)**.** *All pure-strategy equilibria are symmetric.*

As proved in the appendix, Proposition 1 is a consequence of Theorems 11 and 12, demonstrating equilibrium symmetry in the cases $\alpha > 0$ and $\alpha = 0$, respectively. The analysis of the $\alpha = 0$ case (the pure uniform price auction) is essentially identical to the analysis of supply function equilibrium in Klemperer and Meyer (1989). This case must be analyzed separately, since bidding incentives are scaled by the bid-for quantity, and the fundamental theorem of differential equations cannot be applied at $q = 0$. By corollary, all equilibria in the uniform price auction have identical bids, and identical elasticities, at $q = 0$.

When $\alpha > 0$, showing symmetry begins by noting that in any equilibrium, at least two bidders submit the same maximum bid $\bar{b} \equiv b^i(0)$, otherwise at least one of these bidders can reduce her bid without affecting her allocation. Because the optimization problem faced by bidders reduces to solving a differential system, any two bidders with identical maximum bids must be submitting the same bid function.[14] Then if equilibrium is asymmetric, there are at least two distinct maximum bids. Since bidding incentives are smooth in quantity, "high" bidders' bid curves must have a kink at "low" bidders' maximum bid. Then at her maximum bid, the low bidder perceives a discontinuous increase in the elasticity of residual supply. It follows that she can improve her expected utility by slightly increasing her bid, and she is not best responding.

Equilibrium symmetry implies that upon realization of market supply Q, each bidder recieves the same quantity $q = Q/n$. Then because bidders have identical marginal values, equilibrium

[14]This is not true in the uniform price auction, $\alpha = 0$, since the differential equation is degenerate at $q = 0$.

outcomes are efficient, conditional on the distribution of supply.[15] In all equilibria the market price is determined by the (symmetric) bid for per-capita quantity, $p(Q) = b(Q/n)$, and any statement about bids has a simple translation in terms of prices, and vice-versa. Additionally, the probability that a bidder receives less than quantity q, $G^i(q;b)$, is just the probability that the market quantity is at least nq, $G^i(q;b) = F(nq)$. Then the first order conditions in Lemma 1 can be rewritten as

$$(v(q) - b(q)) + \left(\frac{1 - F(nq)}{(n-1) f(nq)}\right) \alpha b_q(q) + \left(\frac{q}{n-1}\right)(1-\alpha) b_q(q) = 0. \tag{4}$$

Solving the differential equation (4) leads to the equilibrium representation in Theorem 1.

Theorem 1 (Equilibrium prices)**.** *In any symmetric equilibrium, there is a constant C such that market clearing prices are*

$$p(Q) = \left(C - \int_0^Q \exp\left(-\int_0^x \tilde{H}(y)\,dy\right) \tilde{H}(x)\,\hat{v}(x)\,dx\right) \exp\left(\int_0^Q \tilde{H}(x)\,dx\right), \tag{5}$$

$$\tilde{H}(z) = \frac{(n-1) f(z)}{n\alpha(1 - F(z)) + (1-\alpha) z f(z)}.$$

Equilibrium bids are

$$b(q) = \left(C - \int_0^{nq} \exp\left(-\int_0^x \tilde{H}(y)\,dy\right) \tilde{H}(x)\,v\left(\frac{1}{n}x\right) dx\right) \exp\left(\int_0^{nq} \tilde{H}(x)\,dx\right).$$

The term $\exp(\int_0^Q \tilde{H}(x)dx)$ is the homogeneous solution to the differential equation implied by (4), and its multiplier C appears in other analyses of divisible-good auctions. In Wang and Zender (2002) it corresponds to the degree of competition in the uniform-price auction; Wang and Zender (2002) and Pycia and Woodward (2019) separately show that it is uniquely determined in the discriminatory auction. In Section 4.1 I show that the set of economically feasible values for C is shrinking in α. Note that $\exp(\int_0^Q \tilde{H}(x)dx)$ depends only on the distribution of quantity, and not on marginal values. Because equilibrium nonuniqueness is related to freedom in the multiplier C, one interpretation is that nonuniqueness is a fundamental feature of non-discriminatory auctions, independent of marginal values.

Remark 1. *Because C scales the homogeneous solution to the best response differential equation, which depends on α, the feasible values of C are not directly comparable across auctions. However, the range of feasible prices is meaningful, and it is feasible to compare "highest" equilibrium bids. The differential equation* (4) *implies that the per-unit margin $v(q) - b(q)$ and the slope of bids $b_q(q)$ will have opposite signs. If $v(q) = b(q)$ ever obtains (with $\alpha > 0$), bids are above marginal values for $q' > q$, which is never optimal.*[16] *Higher equilibrium bids are closer to marginal values, and since C can arbitrarily scale the homogeneous solution, the highest equilibrium bid function must*

[15] Pycia and Woodward (2019) show that when the distribution of supply is set by the seller, the discriminatory auction is fully efficient.

[16] This is proved formally in Lemma 6 in the appendix.

touch marginal values. Then the highest equilibrium bid profile is such that $b(\overline{Q}/n) = v(\overline{Q}/n)$, and the range of feasible prices (and thus values of C) depends on the marginal value for the maximum per-capita quantity.

In light of Remark 1, Theorem 1 implies an upper bound on equilibrium bids, strictly below marginal values.

Corollary 1 (Maximum-bid equilibrium)**.** *The maximum equilibrium price level $\overline{p}$ and bid profile $(\bar{b})_{i=1}^n$ are given by*

$$\overline{p}(Q) = \hat{v}(Q) + \int_Q^{\overline{Q}} \exp\left(\int_x^Q \tilde{H}(y)\,dy\right) \hat{v}_Q(x)\,dx,$$
$$\bar{b}(q) = v(q) + \int_q^{Q^\mu} \exp\left(\int_{nx}^{nq} \tilde{H}(y)\,dy\right) v_q(q)\,dq.$$

The bid profile $(\bar{b})_{i=1}^n$ is the revenue-maximizing equilibrium.

3.1 Polynomial-Lomax model

In the polynomial-Lomax model, marginal values are (order $\bar{k}$) polynomials and supply is distributed according to a negative Lomax distribution. When quantity is distributed according to a negative Lomax distribution, the market price equation (5) is simplified. In particular,

$$\tilde{H}(z) = \frac{(n-1)\lambda}{n\alpha\left(\overline{Q}-z\right) + (1-\alpha)\lambda z}, \quad \exp\left(\int_0^Q \tilde{H}(z)\,dz\right) = \left(1 + \left(\frac{(1-\alpha)\lambda - n\alpha}{n\alpha\overline{Q}}\right)Q\right)^{\frac{(n-1)\lambda}{(1-\alpha)\lambda - n\alpha}}.$$

In the context of the polynomial-Lomax model, there is a natural class of equilibria in which bid functions have the same form as marginal values. I refer to these as *conjugate equilibria*.

Definition 1. *Let marginal values be as in the polynomial-Lomax model with coefficients $(v_k)_{k=0}^{\bar{k}}$. The bid function b is a* conjugate bid function *if there are b_k, $k \in \{0, \ldots, \bar{k}\}$, such that*

$$b(q) = \begin{cases} \sum_{k=0}^{\bar{k}} b_k q^k & \text{if } q \leq \bar{q}_b, \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{q}_b$ is the smallest positive root of $\sum_{k=0}^{\bar{k}} b_k q^k$. The equilibrium bid profile $(b^i)_{i=1}^n$ is a conjugate equilibrium *if each b^i is a conjugate bid function.*

For example, in a model with linear marginal values, equilibrium bids are conjugate if they are linear in quantity; one might expect that bidders will consider bids with the same model by which they construct values. As I show in Theorem 3, when the distribution of supply is negative Lomax, all equilibria are in conjugate bids, plus a potentially degenerate homogeneous term. Because the specification of polynomial marginal values is fairly general, the only restrictive assumption is that supply follows a negative Lomax distribution.

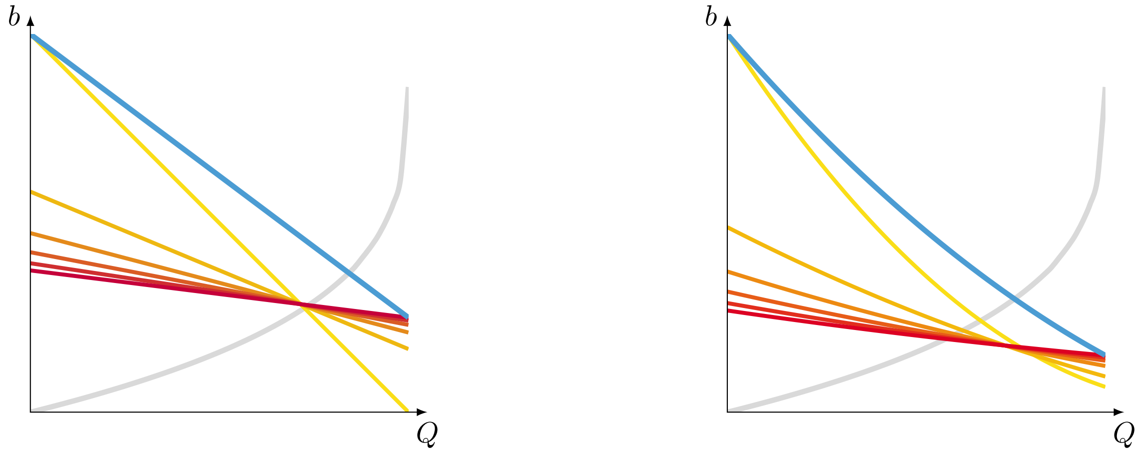

Figure 2: Conjugate bid functions when $\bar{k} = 1$ (left panel) and $\bar{k} = 2$ (right panel). Marginal values are in blue and the CDF of aggregate supply is in gray. Bids in the pure discriminatory auction are in red and bids in the pure uniform-price auction are in yellow. Bids in hybrid auctions, $\alpha \in \{0.2, 0.4, 0.6, 0.8\}$, are colored according to "how discriminatory" the payment rule is.[18]

Theorem 2 (Conjugate prices and bids)**.** *Conjugate prices in the polynomial-Lomax model are given by*

$$\check{p}\left(Q\right) = \sum_{k=1}^{\bar{k}} \check{p}_k Q^k, \quad \check{p}_k = \frac{\left(n-1\right)\lambda}{n\alpha\overline{Q}} \sum_{t=k}^{\bar{k}} \left(\prod_{s=k}^{t} \frac{n\alpha\overline{Q}}{\left(n-1\right)\lambda - \left(\left(1-\alpha\right)\lambda - n\alpha\right)s} \right) \frac{t!v_t}{k!n^t}.$$

Conjugate bids in the polynomial-Lomax model are given by

$$\check{b}\left(q\right) = \sum_{k=1}^{\bar{k}} \check{b}_k q^k, \quad \check{b}_k = \frac{\left(n-1\right)\lambda}{n\alpha\overline{Q}} \sum_{t=k}^{\bar{k}} \left(\prod_{s=k}^{t} \frac{n\alpha\overline{Q}}{\left(n-1\right)\lambda - \left(\left(1-\alpha\right)\lambda - n\alpha\right)s} \right) \frac{t!v_t}{k!n^{t-k}}.$$

The complicated expressions for polynomial coefficients in Theorem 2 mask simple recursive relationships between the coefficients, and the coefficients themselves need not be complicated. For example, in the context of linear marginal values $v(q) = v_0 + v_1 q$, conjugate equilibrium bid coefficients are

$$\check{b}_0 = v_0 + \frac{\alpha\overline{Q}v_1}{\left(n-2\right)\lambda + \left(n+\lambda\right)\alpha}, \quad \check{b}_1 = \frac{\left(n-1\right)\lambda v_1}{\left(n-2\right)\lambda + \left(n+\lambda\right)\alpha}$$
$$\implies \quad \check{b}^{\text{PAB}}\left(q\right) = \left[v_0 + \frac{\overline{Q}v_1}{\left(n-1\right)\lambda + n}\right] + \left[\frac{n-1}{\left(n-1\right)\lambda + n}\right]\lambda v_1 q, \quad \check{b}^{\text{UPA}}\left(q\right) = v_0 + \left[\frac{n-1}{n-2}\right]\lambda v_1 q.$$

As previously noted, the price coefficients of the hybrid auction are not (identical) convex combinations of the discriminatory and uniform price price coefficients.

In the polynomial-Lomax model conjugate bid functions are the basis for all equilibrium be-

[18]In the case of the linear-Lomax model ($\bar{k} = 1$), equilibrium bids intersect at a common $q^{\perp}$. This is particular to the linear-Lomax model, and does not generalize.

havior. Since best response bids are solutions to a differential equation, equilibrium consists of an conjugate bid curve together with a scaling bid curve, representing the differential equation's homogeneous solution.

Theorem 3 (Conjugate basis of equilibrium)**.** *In any equilibrium of the polynomial-Lomax model, there is a constant $C \in \mathbb{R}$ such that the market clearing price is given by*

$$p(Q) = \check{p}(Q) + \left(n\alpha\left(\overline{Q} - Q\right) + (1-\alpha)\lambda Q\right)^{\frac{(n-1)\lambda}{(1-\alpha)\lambda - n\alpha}} C,$$

where $\check{p}$ is the conjugate market clearing price.

Theorem 3 expands known results on the existence of pure-strategy equilibria in uniform-price auctions (Klemperer and Meyer, 1989; Ausubel et al., 2014). With linear marginal values a uniform-price auction must have at least three bidders to admit a pure-strategy equilibrium. This follows from Theorems 2 and 3, since in the pure uniform price case, $\alpha = 0$, there is a term in the denominator of the conjugate price coefficient equation which is equal to $(n-1) - k$. When the order of the polynomial marginal value function, $\bar{k}$, is weakly greater than $n-1$, there is $k \in \{0, \ldots, \bar{k}\}$ such that $(n-1) - k = 0$, implying that there is a price coefficient which is infinite. Since conjugate bid functions are the basis for equilibrium behavior, it follows that whenever the number of bidders is below the order of the marginal value polynomial (minus one) there is no pure-strategy equilibrium in the uniform-price auction. An immediate corollary is that when marginal values are linear (the order of the polynomial is one) there must be at least three bidders in the auction for a pure-strategy equilibrium to exist. Since uniform price equilibria are ex post, and the differential system (4) does not depend on F, the following corollary applies to all uniform price auctions with random supply supported on $[0, \overline{Q}]$, and not only those with a negative Lomax distribution of supply.

Corollary 2 (Nonexistence of uniform price auction equilibrium)**.** *If $\bar{k}$, the order of the marginal value polynomial $\sum_{k=0}^{\bar{k}} v_k q^k$, is at least $n-1$, there is no equilibrium of the uniform price auction.*

Finally, as noted in Remark 1 and Corollary 1, the highest-bid equilibrium is determined by setting the market price for the maximum quantity to (aggregate) marginal value for the maximum quantity, $\overline{p}(\overline{Q}) = \hat{v}(\overline{Q})$; conjugate and maximum prices are illustrated in Figure 4.

Corollary 3 (Maximum-bid equilibrium$'$)**.** *In the polynomial-Lomax model, the highest equilibrium market clearing price function is given by*

$$\overline{p}(Q) = \check{p}(Q) + \left(\frac{n\alpha}{(1-\alpha)\lambda} + \left(1 - \frac{n\alpha}{(1-\alpha)\lambda}\right)\frac{Q}{\overline{Q}}\right)^{\frac{(n-1)\lambda}{(1-\alpha)\lambda - n\alpha}} \left(\hat{v}\left(\overline{Q}\right) - \check{p}\left(\overline{Q}\right)\right),$$

where $\check{p}$ is the conjugate market clearing price.

For certain parameterizations it is the case that $\check{p}(\overline{Q}) > \hat{v}(\overline{Q})$, and therefore $\overline{p}(Q) < \check{p}(Q)$. When this is the case there does not exist a conjugate equilibrium, but conjugate bid functions are

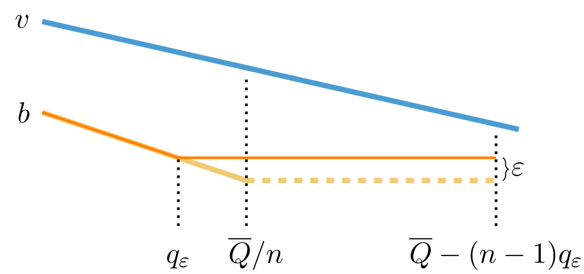

Figure 3: The deviation used to establish the necessary condition for a particular bid function to be sustainable in equilibrium. This deviation cannot be profitable, establishing an upper bound on $v(\overline{Q}/n) - p(\overline{Q})$.

still the basis for equilibrium behavior (Theorem 3): the homogeneous scaling term simply needs to be sufficiently negative.

4 Equilibrium properties

The equilibrium expressions developed in Section 3 imply some useful comparative statics with regard to hybridization α. First, equilibrium is "increasingly unique" as the hybrid auction becomes discriminatory. Second, in large markets, the incentives in the hybrid auction are hybridization-scaled versions of discriminatory auction incentives. Third, in conjugate equilibrium of the linear-Lomax model, revenue is strictly increasing in the extent of price discrimination, and the unique equilibrium of the discriminatory auction raises more revenue than the maximum-bid equlbirium of the uniform price auction.

4.1 Equilibrium uniqueness

Equilibrium uniqueness in discriminatory auctions has been established by Wang and Zender (2002) and Pycia and Woodward (2019), while Klemperer and Meyer (1989) and Back and Zender (1993) show that the uniform price mechanism can sustain a broad multiplicity of equilibria. Because equilibrium bid functions are the solution to a differential equation, equilibrium nonuniqueness is closely related to the set of economically feasible initial conditions. In this section I develop a condition for feasibility which is easier to satisfy the less discriminatory is the hybrid auction. It follows that uniform price auctions may admit more equilibria than hybrid auctions, which may admit more equilibria than discriminatory auctions.

The feasibility condition is derived from the analysis of small upward deviations in bid near the maximum per-capita quantity $\overline{Q}/n$ (see Figure 3).[19] In particular, a bidder could impose a floor on her own bid, adjusting it so it is never less than $b(\overline{Q}/n) + \varepsilon$.[20] Doing so will increase her allocation

[19] This is, locally, a more powerful tool than the Jacobi equation from the calculus of variations, which requires only that a certain function lack zeros. Here, I gain leverage from understanding the sign of a particular function of the market price and model fundamentals.

[20] The model in the main text specifies that bid functions are strictly decreasing, for ease of exposition. The analysis

when the realization of Q is large, and will also increase her payment. This deviation cannot be profitable in equilibrium, implying a bound on the distance between bid and marginal value. This places a natural limit on the set of permissible equilibrium market-clearing price functions.

Remark 2. *The endpoint condition considered in Lemma 2 is one of many possible conditions for equilibrium existence. It can be shown that any similar deviation considered at* $q \leq Q^\mu \equiv \overline{Q}/n$ *will yield the same conditions as the first order condition in Lemma 1. Furthermore, equilibrium bids are not specified for infeasible quantities* $q > Q^\mu$*, and bidders can submit bids which are equal to* $b(Q^\mu)$ *for all infeasible* q*. In this case, a downward deviation near* $q = Q^\mu$ *strictly reduces quantity allocations, and is never profitable. On the other hand, upward deviations near* $q = Q^\mu$ *have additional analytical power beyond the established first order conditions, because optimization using the calculus of variations presumes that endpoint conditions are fixed, where in divisible-good auctions they are flexible and must be derived from the theory.*

Because market prices are strictly decreasing in quantity Q, the necessary condition for the price at the maximum quantity, $p(\overline{Q})$, can be framed as the minimum price feasible in equilibrium.

Lemma 2 (Equilibrium necessary condition)**.** *A necessary condition for a market price function* p *to represent an equilibrium is*

$$\frac{1-2\alpha}{1-\alpha}\left(p\left(\overline{Q}\right)-\hat{v}\left(\overline{Q}\right)\right)+\overline{Q}\hat{v}_Q\left(\overline{Q}\right)\leq 0. \tag{6}$$

Corollary 4 (Equilibrium range of minimum prices)**.** *In equilibrium, when the hybridization is sufficiently discriminatory* $(\alpha > 1/2)$*, the minimum market clearing price must satisfy*

$$p\left(\overline{Q}\right)\in\left[\hat{v}\left(\overline{Q}\right)-\frac{1-\alpha}{1-2\alpha}\overline{Q}\hat{v}_Q\left(\overline{Q}\right),\hat{v}\left(\overline{Q}\right)\right]_+.$$

When the hybridization is sufficiently non-discriminatory $(\alpha \leq 1/2)$*, the minimum market clearing price must satisfy* $p(\overline{Q}) \in [0, \hat{v}(\overline{Q})]$*.*

Lemma 2 places only a very weak restriction on the equilibrium price function. It does not guarantee that a particular solution is an equilibrium, only that the response cannot be locally improved-upon from deviations near $\overline{Q}/n$. Note that, in the pure discriminatory auction ($\alpha = 1$), Lemma 2 implies $p(\overline{Q}) = \hat{v}(\overline{Q})$, as identified in Pycia and Woodward (2019). Lemma 2 is sufficient to derive monotonicity of the set of feasible initial conditions, and therefore on the set of equilibrium bid functions.

Lemma 3 (α-monotonicity of necessary condition)**.** *Suppose that* p *is a solution to the equilibrium market-price equation for randomization* α*, and satisfies the necessary condition of Lemma 2. Then*

in the appendix shows that equilibrium bid functions must be strictly decreasing and continuous, so this assumption is without loss when considering equilibrium outcomes. However, bid-floor deviations are not strictly decreasing, and thus are disallowed by the main-text model. The constructed deviations are consistent with the analysis in the appendix, and can be approximated in the main-text model by bids with arbitrarily-small (but still strictly negative) slopes.

for any $\alpha' < \alpha$, there is a solution to the equilibrium market-price equation p' with $p'(\overline{Q}) = p(\overline{Q})$ which satisfies the necessary condition of Lemma 2.

Lemma 3 follows from the observation that, holding fixed $p(\overline{Q}) < \hat{v}(\overline{Q})$, the left-hand side of inequality (6) is monotonically increasing in α. Then a large enough increase in hybridization α will move the left-hand side from negative to positive, violating condition (6). Given hybridization α, let $\mathrm{P}(\alpha)$ be the set of endpoints to valid solutions to the market-clearing equation, so that price is everywhere-positive, its implied bids are below the agents' value functions, and satisfies the inequality of Lemma 2. This construction provides the following result.

Corollary 5 (Decreasing multiplicity). *Let $\mathrm{P}(\alpha)$ be the set of endpoints to valid solutions to the market clearing equation, so that (i) price is weakly positive, (ii) implied bids are below agents' marginal values, and (iii) inequality* (6) *is satisfied. Then* P *is ordered by reverse-inclusion: $\alpha' < \alpha$ implies* $\mathrm{P}(\alpha') \supseteq \mathrm{P}(\alpha)$.

The set of permissible market price functions, according to endpoint conditions, is increasing as α falls, in the sense that the set of available endpoint prices is growing. A stronger claim is not immediately feasible, as α fundamentally alters the shape of the market price equation. Inclusion-monotonicity of P with respect to α provides evidence that the multiplicity of equilibria in the uniform price auction and the uniqueness of equilibrium in the discriminatory auction are not knife-edge cases obtained by mechanism degeneracy, but are part of a continuum of feasible initial conditions. Indeed, the set of permissible intial conditions for the market-clearing price function is shrinking smoothly as the mechanism moves from uniform price to discriminatory.

Perhaps surprisingly, inequality (6) does not depend on the number of bidders n. While equilibrium bids depend on the number of bidders, the set of feasible endpoint conditions does not. Fixing an initial condition $p(\overline{Q})$, the market clearing price equation (4) implies that $p_Q(\overline{Q})$ grows without bound as n increases. As the number of bidders grows large, bids with nontrivial homogeneous terms become inelastic at $\overline{Q}$, and the incentive to deviate upward is lessened. In equilibrium, falling per-bidder elasticity exactly offsets the increase in the number of bidders. I explore this effect further in Section 4.2 below.

Finally, while Corollary 5 shows that the set of feasible prices $p(\overline{Q})$ is shrinking in α, the range of feasible prices $p(Q)$ for any quantity Q is increasing in Q. Let $R^n(Q)$ be the range of feasible equilibrium prices for quantity Q when there are n bidders. Because different equilibrium bid functions cannot intersect, Corollary 4 implies that there is a highest equilibrium bid function $\bar{b}$ and a lowest equilibrium bid function $\underline{b}$, where

$$\bar{b}\left(Q^{\mu}\right) = v\left(Q^{\mu}\right), \quad \underline{b}\left(Q^{\mu}\right) = \max\left\{0, v\left(Q^{\mu}\right) - \frac{1-\alpha}{1-2\alpha}\left(Q^{\mu}\right) v_q\left(Q^{\mu}\right)\right\}.$$

Then for any Q, $R^n(nq) = \bar{b}(q) - \underline{b}(q)$.

Proposition 2 (Feasible price range increasing in Q). *The range of feasible equilibrium prices, $R^n(Q)$, is weakly increasing in Q.*

Proof. Lemma 2 implies that $R^n(\overline{Q})$ is constant in n. From the equilibrium first order conditions,

$$\overline{b}_q(q) = -\frac{(n-1)\left(v(q) - \overline{b}(q)\right) f(nq)}{\alpha(1 - F(nq)) + (1-\alpha) q f(nq)}, \quad \underline{b}_q(q) = -\frac{(n-1)\left(v(q) - \underline{b}(q)\right) f(nq)}{\alpha(1 - F(nq)) + (1-\alpha) q f(nq)}.$$

Then

$$nR^n_Q(nq) = \overline{b}_q(q) - \underline{b}_q(q) = \frac{(n-1) f(nq)}{\alpha(1 - F(nq)) + (1-\alpha) q f(nq)} R^n(nq) \geq 0.$$

□

Proposition 2 relates equilibrium uniqueness to the range of observable market prices, and is illustrated in Figure 4. When the market quantity Q is small, the range of feasible market prices will be small; when the market quantity Q is large, the range of feasible market prices will be (comparatively) large. This is particularly apparent at the extreme, pure auction implementations. In a discriminatory auction, there is a unique feasible initial condition $p(\overline{Q})$; Proposition 2 then implies that there is a unique equilibrium. At the other extreme, the set of feasible initial conditions P is maximized in a uniform price auction (Corollary 5), but in a uniform price auction the zero-quantity price is uniquely determined, $p(0) = v(0)$.

It is straightforward to see that there is no analogue of Proposition 2 with regard to changes in α: when $\alpha = 1$, there is a unique equilibrium, and when $\alpha = 0$ there is a unique price for quantity $Q = 0$. Because hybrid auctions $\alpha \in (0,1)$ admit a range of prices for quantity $Q = 0$, it follows that the range of equilibrium prices is nonmonotone in α. This is visible in Figure 4, where for Q small, the range of feasible prices in the hybrid auction strictly exceeds the range of feasible prices in discriminatory and uniform price auctions. However, it is possible to derive comparative statics on the whole set of observable prices. Let $\mathbf{P}(\alpha; n)$ be the set of equilibrium market clearing prices in hybridization α when there are n bidders,

$$\mathbf{P}(\alpha; n) \equiv \left\{ b(q) : \ q \in [0, Q^\mu], \ b \text{ solves (4), and } b(Q^\mu) \in \mathrm{P}(\alpha) \right\}.$$

The set of equilibrium market clearing prices is shrinking in α.

Theorem 4 (Feasible price space decreasing in discrimination)**.** *Let $\alpha < \alpha'$. Then $\mathbf{P}(\alpha; n) \subseteq \mathbf{P}(\alpha'; n)$.*

Proof. Corollary 5 shows that the minimum feasible market clearing price is increasing in α. Since in any equilibrium bids are continuous, it is sufficient to show that in the maximum-bid equilibrium, $\overline{b}^\alpha(0)$ is decreasing in α. Suppose otherwise; then there are α and α', $\alpha < \alpha'$, such that $\overline{b}^\alpha(0) < \overline{b}^{\alpha'}(0)$. For any q such that $\overline{b}^\alpha(q) < \overline{b}^{\alpha'}(q)$, it must be that $|\overline{b}^\alpha(q)| > |\overline{b}^{\alpha'}(q)$, and it follows that $\overline{b}^\alpha$ and $\overline{b}^{\alpha'}$ cannot cross. This contradicts the construction of maximum-bid equilibrium, where $\overline{b}(Q^\mu) = v(Q^\mu)$. □

Theorem 4 suggests that sellers' outcomes are less certain in uniform price auctions than in discriminatory auctions. The range of feasible equilibrium prices is decreasing in hybridization

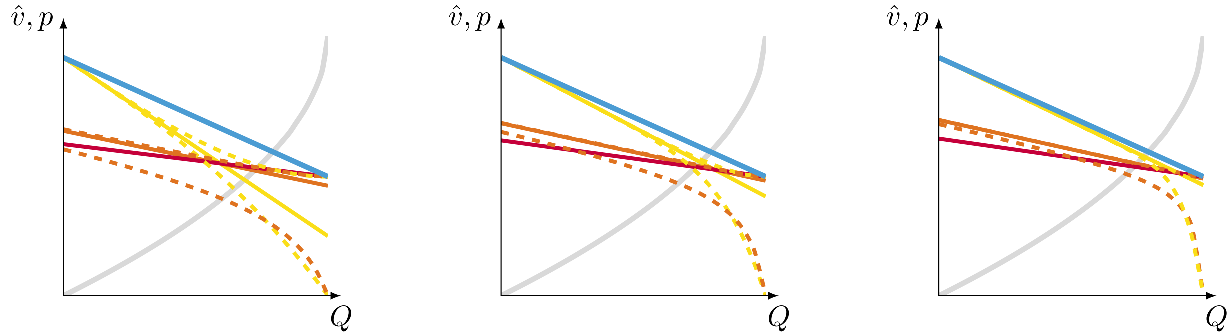

Figure 4: Conjugate prices, and upper and lower bounds, in the linear-Lomax model for $n \in \{4, 8, 16\}$ (left to right). Uniform price prices are in yellow, discriminatory prices are in red, and hybrid prices ($\alpha = 0.5$) are in orange. As the number of bidders becomes large, the range of feasible initial conditions $p(\overline{Q})$ does not change, but the conjugate price approaches $\hat{v}(\overline{Q})$ and the slope of the lower bound of prices goes to infinity. For any quantity Q, the range of feasible prices, as measured by the spread, shrinks when the number of bidders increases.

α, and therefore there is a wider range of observed transfers in uniform price auctions than in discriminatory auctions. This occurs because bids are less elastic in uniform price auctions, and because there is a broader range of equilibria.

The results of this subsection offer three perspectives on nonuniqueness. First, the set of feasible equilibrium initial conditions — and therefore, in a natural sense, the size of the equilibrium set — is decreasing in price discrimination. Second, the range of feasible prices is highest for large quantities. Since equilibrium bids for the maximum per capita quantity are less elastic the less discriminatory is the auction, when the distribution of supply is skewed towards large quantities the set of observed prices will be larger in less-discriminatory auctions. Third, the entire set of feasible equilibrium prices is shrinking in price discrimination. In total, I observe that equilibrium outcomes are more certain the more discriminatory is the hybridization.

4.2 Large markets

I now consider the case of an auction with a large number of bidders. If quantity is held constant while the number of participants increases, in the limit no agent can receive a strictly positive quantity. Then the unique equilibrium prediction is truthful reporting at $q = 0$, independent of the auction implemented. It is natural then to consider the large-market limit as per-capita quantity $Q^\mu \equiv \overline{Q}/n$ is held constant. In this limit, I show that equilibrium incentives are simply scaled versions of discriminatory auction incentives, regardless of the auction implemented. To simplify notation, let F^μ denote the distribution of per-capita quantity, so that for any number of bidders n, $F^\mu(Q) = F^n(nQ)$.

Although the endpoint condition identified in Lemma 2 does not depend on the number of bidders n, the range of feasible equilibrium prices shrinks as the market becomes large.

Proposition 3 (Feasible price range decreasing in n)**.** *For all $q \in [0, Q^{\mu}]$, $R^n(nq)$ is decreasing in n.*

Proof. Lemma 2 implies that $R^n(nQ^{\mu})$ is constant in n. As in the proof of Proposition 2,

$$
\begin{aligned}
nR_Q^n(nq) = \bar{b}_q(q) - \underline{b}_q(q) &= \frac{(n-1) f(nq)}{\alpha(1 - F(nq)) + (1-\alpha) q f(nq)} R^n(nq) \\
&= \frac{(n-1) f^{\mu}(q)}{n\alpha(1 - F^{\mu}(q)) + (1-\alpha) q f^{\mu}(q)} R^n(nq) \equiv H^n(q) R^n(nq).
\end{aligned}
$$

The ratio H^n is increasing in n. Then letting $n' > n$, if q is such that $R^{n'}(n'q) = R^n(nq)$, then $n'R_Q^{n'}(n'q) > nR_Q^n(nq)$. It follows that R^n and $R^{n'}$ may cross only once. Since they meet when $q = Q^{\mu}$, it follows that $R^{n'}(n'q) \leq R^n(nq)$ for all $q \in [0, Q^{\mu}]$, and this inequality is strict for all $q < Q^{\mu}$. □

Surprisingly, while the range of feasible prices for the given quantity is *decreasing* in the number of bidders n, the space of feasible prices is *increasing* in n. Per-quantity price uniqueness increases with competition, but so too does the slope of equilibrium bids. Since the endpoint condition identified in Lemma 2 is independent of the number of bidders, it follows that the range of initial prices $p(0)$ is increasing in n, and therefore so is the space of feasible prices.

Theorem 5 (Feasible price space increasing in n)**.** *Let $n < n'$. Then $\mathbf{P}(\alpha; n) \subseteq \mathbf{P}(\alpha; n')$.*

Proof. Recall the market clearing first order condition,

$$
b_q(q) = -\frac{(n-1)(v(q) - b(q)) f^{\mu}(q)}{n\alpha(1 - F^{\mu}(q)) + (1-\alpha) q f^{\mu}(q)} = H^n(q)(v(q) - b(q)).
$$

Since $H^n(q)$ is increasing in n, if $v(q) - b^n(q) = v(q) - b^{n+1}(q)$, it follows that $|b_q^n(q)| < |b_q^{n'}(q)|$, and bid curves can intersect at most once. For all $\alpha < 1$, the fact that $\mathrm{P}(\alpha)$ is independent of n implies that $\mathbf{P}(\alpha; n) \subseteq \mathbf{P}(\alpha; n')$. For the discriminatory auction, $\alpha = 1$, equilibrium is unique, and limiting arguments for $q \nearrow Q^{\mu}$ imply that $b^n(q) < b^{n'}(q)$. □

The space of feasible prices is increasing in n, but as n increases low prices become less and less likely. Low prices arise from equilibrium selection, and (as I show below) low-price equilibria have low elasticities when prices are low. In the limit, $n \nearrow \infty$, these low-probability prices become zero-probability prices, and the feasible price space discontinuously shrinks.

When the number of bidders is large, uniform price demand-reduction incentives (proportional to $q/(n-1)$) go to zero, and discriminatory auction demand-reduction incentives dominate.With a large number of bidders and elastic demand, a small deviation in bid will have a dramatic effect on resulting allocations. Since, in a uniform price auction, the bid for a particular quantity affects payment only for this quantity, bids must equal marginal values. In a discriminatory auction it remains true that small deviations will have dramatic effects on allocations, but the bid for quantity q is paid whenever $Q > nq$. The first order conditions given in Lemma 1 relate the margin $v(q) - b(q)$

to $b_q(q)$, the slope of the bid function at q. In the large-market limit, there is no incentive to bid a positive margin in a uniform price auction, while such an incentive remains in a discriminatory auction, and the resulting first order conditions take the form of a scaled discriminatory auction.

Theorem 6 (Scaled discriminatory incentives in large markets)**.** *In the large-market limit, equilibrium first-order conditions are given by*

$$-\left(v\left(q\right)-b\left(q\right)\right)=\alpha\frac{1-F^{\mu}\left(q\right)}{f^{\mu}\left(q\right)}b_{q}\left(q\right). \tag{7}$$

Theorem 6 does not imply that uniform price incentives are not a part of the large market analysis. Large-market incentives appear to be discriminatory incentives, but the scaling is closely related to uniform price incentives. In a uniform price auction, the first order condition is equal to the margin, $v(q) - b(q)$, and in the hybrid auction this margin is obtained at weight $1 - \alpha$. Then it is the consideration of this term that induces the scaling factor in (7).

As in the finite-n model, the differential equation in (7) has a closed form solution. This equilibrium is unique, implying that Proposition 3 may be interpreted as indicating a smooth transition from nonuniqueness to uniqueness as markets become large.

Theorem 7. *In the large-market limit, equilibrium bids are given by*

$$b\left(q\right)=\int_{q}^{Q^{\mu}}v\left(x\right)dF^{\alpha,q}\left(x\right),\quad F^{\alpha,q}\left(x\right)=1-\left(\frac{1-F^{\mu}\left(x\right)}{1-F^{\mu}\left(q\right)}\right)^{\frac{1}{\alpha}}.$$

This equilibrium is unique.

Price discrimination and market size have similar effects on equilibrium uniqueness: price discrimination reduces the set of initial conditions $p(\overline{Q})$, while market size reduces the set of feasible elasticities, conditional on $p(Q) \leq \hat{v}(Q)$. In either case, equilibrium uniqueness obtains in the limit, either $\alpha = 1$ or $n \nearrow \infty$. When $\alpha < 1$, large-market equilibrium nonuniqueness arises from the homogeneous solution to the market clearing equation. The homogeneous term is derived from uniform price incentives. Then as market grow large and uniform price incentives are diminished (Theorem 6), equilibrium becomes unique.

Remark 3. *In the linear-Lomax model, the slope of conjugate bids is $b_1 = ((n-1)\lambda v_1)/((n-2)\lambda + (n+\lambda)\alpha)$. As n becomes large, this approaches $b_1^{\infty} \equiv \lambda v_1/(\lambda + \alpha)$, which is finite. However, the slope of the homogeneous term, evaluated at Q^{μ}, is*

$$b_{q}^{h}\equiv\left(n-1\right)\lambda\left(\left(1-\alpha\right)\lambda Q^{\mu}\right)^{\frac{(n-2)\lambda+(n+\lambda)\alpha}{(1-\alpha)\lambda-n\alpha}}.$$

For all $\alpha > 0$, this slope goes to $+\infty$ as n becomes large. This is depicted in Figure 4.

The equilibrium bid representation in Theorem 7 demonstrates a number of interesting features. First, bids are truthful for the maximum per-capita allocation Q^{μ}, $b(Q^{\mu}) = v(Q^{\mu})$. In light of

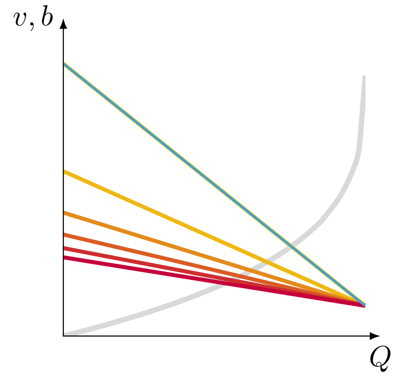

Figure 5: Equilibrium bids in the linear-Lomax model for $n \nearrow \infty$. Because equilibrium is unique, equilibrium bids are conjugate bids, without any homogeneous term. Uniform price auction bids are truthful, and are therefore hidden behind the (blue) marginal value curve.

Lemma 4 this implies that no higher bid function is sustainable in equilibrium. As mentioned above, uniqueness follows from showing that the homogeneous solution to the first-order conditions diverges as $n \nearrow \infty$.

Second, bids are truthful in the uniform price auction, $\alpha = 0$.[21] In all cases, the bid for quantity q is the expected value of marginal values for larger quantities, taken with respect to a reweighted distribution $F^{\alpha,q}$; when $\alpha = 0$, this distribution is degenerate at q. With an infinite number of opponents, bidder i cannot affect the market-clearing price, only her allocation. If she is bidding truthfully she can obtain a larger quantity only at a negative margin, and if she obtains a smaller quantity she loses positive marginal gains. Neither is utility-improving and thus truthful reporting emerges.

Third, for all $q < Q^\mu$, bids are strictly decreasing in α,

$$b_\alpha(q) = \int_q^{Q^\mu} \underbrace{v_q(q)}_{<0} \underbrace{\left(-\frac{1}{\alpha^2}\right)}_{<0} \underbrace{\left(\frac{1 - F^\mu(x)}{1 - F^\mu(q)}\right)^{\frac{1}{\alpha}}}_{\geq 0} \underbrace{\ln \frac{1 - F^\mu(x)}{1 - F^\mu(q)}}_{\leq 0} dx.$$

This contrasts the finite-n case, where under different hybridizations, equilibrium bids may cross. Since bids in the large-market uniform price auction are truthful, it follows that the more a randomized auction resembles a uniform price auction, the more truthful are its equilibrium bids. Figure 5 illustrates all three effects.

The closed-form expression for equilibrium bids in a large market makes the following result immediate.

Proposition 4 (Large market revenue equivalence)**.** *In large markets, equilibrium per-capita revenue is independent of the hybridization* α*.*

[21] Swinkels (2001) obtains large-market truthfulness in multi-unit auctions.

Proof. Equilibrium expected per-capita revenue is

$$\begin{aligned}
&\mathbb{E}_{F^\mu}\left[\alpha\int_0^q b(x)\,dx+(1-\alpha)\,qb(q)\right]\\
&=\int_0^{Q^\mu}\alpha b(q)\,(1-F^\mu(q))+(1-\alpha)\,qb(q)\,f^\mu(q)\,dq\\
&=\int_0^{Q^\mu}\left(\alpha\,(1-F^\mu(q))+(1-\alpha)\,qf^\mu(q)\right)\int_q^{Q^\mu}v(x)\,dF^{\alpha,q}(x)\\
&=\int_0^{Q^\mu}\alpha\frac{d}{dy}\left[y\,(1-F^\mu(y))^{1-\frac{1}{\alpha}}\right]\Big|_{y=q}(1-F^\mu(q))^{\frac{1}{\alpha}}\int_q^{Q^\mu}\frac{1}{\alpha}v(x)\left(\frac{1-F^\mu(x)}{1-F^\mu(Q)}\right)^{\frac{1}{\alpha}-1}\frac{f^\mu(x)}{1-F^\mu(q)}dxdq\\
&=\int_0^{Q^\mu}q\,(1-F^\mu(q))^{1-\frac{1}{\alpha}}\,v(q)\,(1-F^\mu(q))^{\frac{1}{\alpha}-1}f(q)\,dq=\mathbb{E}_{F^\mu}\left[qv(q)\right].
\end{aligned}$$

Then in large markets, all hybrid auctions generate revenue identical to pure uniform price auctions. □

Revenue equivalence is a standard property of single-unit auctions, but does not generalize to the multi-unit context.[22] Swinkels (2001) shows that in large multi-unit auctions, equilibrium revenue approaches the expected maginal value in an efficient allocation. Proposition 4 has a similar competitive intuition: in a large market in the divisible-good model, equilibrium utility is the expected utility in a Bertrand model with known supply. Finally, Pycia and Woodward (2019) show that divisible-good auctions are revenue-equivalent when the seller has discretion over the distribution of supply.

Although large markets generate identical expected revenue, independent of hybridization, in markets with a finite number of bidders, expected revenue will depend on the hybridization α. I explore this effect below.

4.3 Expected revenue

Revenue comparison is hampered by the complex effect of hybridization on equilibrium bids. Nonetheless, in the conjugate equilibrium of the linear-Lomax model, equilibrium expected revenue is strictly increasing in price discrimination. Since revenue is strictly increasing as the auction becomes increasingly discriminatory, the resulting revenue ranking will hold in a neighborhood of the linear-Lomax model.

Theorem 8 (Equilibrium revenue)**.** *In conjugate equilibrium of the polynomial-Lomax model, expected revenue is*

$$\mathbb{E}[\pi]=\sum_{k=0}^{\bar{k}}\left(1-\frac{\alpha k}{k+1}\right)p_k\mathbb{E}\left[Q^{k+1}\right].$$

[22]**?** shows that the "efficient allocation" condition for revenue equivalence identified by Myerson (1981) and Riley and Samuelson (1981) generalizes to the multi-unit context. However, this condition is not generally satisfied by standard multi-unit auction formats (Ausubel et al., 2014).

Applying closed-form expressions for the moments of the Lomax distribution gives the following Corollary.

Corollary 6 (Equilibrium revenue′)**.** *In conjugate equilibrium of the polynomial-Lomax model, expected revenue is*

$$\mathbb{E}\left[\pi\right] = \sum_{k=0}^{\bar{k}} \left(1 - \frac{\alpha k}{k+1}\right) p_k \prod_{t=1}^{k+1} \frac{t\overline{Q}}{\lambda + t}.$$

When marginal values are linear, Corollary 6 implies that equilibrium revenue is strictly increasing in hybridization α.

Theorem 9 (Revenue increasing in α)**.** *In conjugate equilibrium of the linear-Lomax model, per capita revenue is strictly increasing in α.*

The algebraic proof of Theorem 9 suggests that revenue is strictly increasing in α, including when $\alpha > 1$. That is, apparent revenue could be even higher in a "double-discriminatory auction with a rebate." However, in conjugate equilibrium of the linear-Lomax model the minimum price $p(\overline{Q})$ is increasing in hybridization α. Since $p^{\text{PAB}}(\overline{Q}) = \hat{v}(\overline{Q})$, this implies that when $\alpha > 1$, $p(\overline{Q}) > \hat{v}(\overline{Q})$, which cannot be the case in equilibrium. A formal analysis of super-discriminatory auctions therefore requires specifying the homogeneous term of equilibrium bids. Maximum-bid equilibria are a natural target for analysis.

Even in normal hybrid auctions ($\alpha \in [0,1]$), the linear-Lomax model may not admit a conjugate equilibrium, and even when it does so, this equilibrium will not in general maximize the seller's revenue. Constraining attention to pure auction implementation, I show that the unique equilibrium of the discriminatory auction strictly revenue-dominates the maximum-bid equilibrium of the uniform price auction (as defined in Corollary 3, and therefore strictly revenue dominates all equilibria of the uniform price auction.

Theorem 10 (Discriminatory dominates uniform price)**.** *In the linear-Lomax model, the unique equilibrium of the discriminatory auction strictly revenue dominates all equilibria of the uniform price auction.*

The proof of Theorem 9 shows a strictly positive revenue difference between the discriminatory auction and the uniform price auction, for any finite n. However, per capita revenue difference goes to 0 as n becomes large;[23] the proof of Theorem 9 does not contradict the large market revenue equivalence result in Proposition 4.

5 Conclusion

This paper considers a model of divisible-good auctions in which the allocation rule is a convex combination of discriminatory and uniform price allocation rules. I show that all equilibria are symmetric, and derive a closed form expression for equilibrium bids and market clearing prices.

[23]The proportionality chain applied in the proof of Theorem 9 simplifies analysis by removing a series of irrelevant constants. When these constants are left in, it is apparent that the revenue difference is increasing in n.

Uniform price auctions are known to admit multiple equilibria, while discriminatory auctions admit unique equilibria. I show that equilibrium multiplicity is smoothly related to price discrimination: the more discriminatory is an auction, the smaller the range of equilibrium prices it can sustain. By corollary, the range of equilibrium prices is increasing in the number of bidders, regardless of the hybridization. However, many of these prices are obtained on a shrinking set of quantities, and in the large-market limit equilibrium is unique.

In parameterized models explicit revenue comparison is possible. I define a bid function to be conjugate it takes the same functional form as marginal values, and show that, when the distribution of supply is negative Lomax, conjugate bids form the basis of all equilibria. In conjugate equilibrium with linear marginal values, expected revenue is strictly increasing in price discrimination. This is not because the seller is discriminating against fixed bid curves, but is instead net of discrimination-induced changes in equilibrium bids. Constraining attention to revenue-maximizing equilibria, the discriminatory auction strictly outperforms the uniform price auction.

In total, these results suggest that discriminatory auctions may generate strictly better seller outcomes than uniform price auctions. However, when markets are large this difference may be minimal. This is in line with known results on the theory of divisible-good auctions — Pycia and Woodward (2019) show that when the seller can affect the distribution of supply, the discriminatory auctions outperform uniform price auctions — as well empirical analyses of multi-unit auctions — Hortaçsu et al. (2018) show that auction format has a neglibile effect on counterfactual outcomes.

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A Equilibrium computations

A.1 Generic equilibrium first-order conditions

Proof of Theorem 1. Let G^i be the distribution of bidder i's allocation. Bidder i's expected utility can be written as

$$\mathbb{E}\left[u\left(b^i, b^{-i}\right)\right] = \int_0^{\overline{Q}} \int_0^q v\left(x\right) - \alpha b^i\left(x\right) dx - \left(1-\alpha\right) q b^i\left(q\right) dG^i\left(q; b^i\right).$$

Integration by parts removes the double-integration, yielding

$$\mathbb{E}\left[u\left(b^i, b^{-i}\right)\right] = \int_0^{\overline{Q}} \left(v\left(q\right) - b^i\left(q\right) - \left(1-\alpha\right) q b_q^i\left(q\right)\right)\left(1 - G^i\left(q; b^i\right)\right) dq.$$

Applying the calculus of variations yields that for all q,

$$-\left(1 - G^i\left(q; b^i\right)\right) - \left(v\left(q\right) - b^i\left(q\right) - \left(1-\alpha\right) q b_q^i\left(q\right)\right) G_b^i\left(q; b\right) = \frac{d}{dq}\left[-\left(1-\alpha\right) q\left(1 - G^i\left(q; b^i\right)\right)\right].$$

Expanding the derivative yields

$$-\left(v\left(q\right) - b^i\left(q\right)\right) G_b^i\left(q; b^i\right) = \alpha\left(1 - G\left(q; b^i\right)\right) + \left(1-\alpha\right) q G_q^i\left(q; b^i\right). \quad (8)$$

In a symmetric equilibrium all of bidder i's opponents will submit the same bid function b. Without loss of generality (see Lemma 5) this bid function has a well-defined inverse φ. By construction, $\varphi_p(b^i(q)) = 1/b_q^i(q)$. Then

$$G^i\left(q; b^i\right) = F\left(q + \left(n-1\right)\varphi\left(b^i\left(q\right)\right)\right).$$

It follows that

$$G_q^i\left(q; b^i\right) = f\left(q + (n-1)\,\varphi\left(b^i\left(q\right)\right)\right),$$
$$G_b^i\left(q; b^i\right) = (n-1)\,\varphi_p\left(b^i\left(q\right)\right) f\left(q + (n-1)\,\varphi\left(b^i\left(q\right)\right)\right).$$

In a symmetric equilibrium it must be that $\varphi(b^i(q)) = q$. Then

$$G^i\left(q; b\right) = F\left(nq\right), \quad G_q^i\left(q; b\right) = f\left(nq\right), \quad G_b^i\left(q; b\right) = \frac{n-1}{b_q\left(q\right)} f\left(nq\right).$$

Substituting into equation (8) yields

$$-\left(v\left(q\right) - b\left(q\right)\right) f\left(nq\right) = \frac{1}{n-1}\left(\alpha\left(1 - F\left(nq\right)\right) + \left(1-\alpha\right) q f\left(nq\right)\right) b_q\left(q\right).$$

By market clearing it must be that $p(Q) = b(Q/n)$. Then

$$-\left(v\left(q\right) - p\left(nq\right)\right) f\left(nq\right) = \frac{n}{n-1}\left(\alpha\left(1 - F\left(nq\right)\right) + \left(1-\alpha\right) q f\left(nq\right)\right) p_Q\left(nq\right)$$
$$\implies \quad -\left(\hat{v}\left(Q\right) - p\left(Q\right)\right) f\left(Q\right) = \frac{1}{n-1}\left(n\alpha\left(1 - F\left(Q\right)\right) + \left(1-\alpha\right) Q f\left(Q\right)\right) p_Q\left(Q\right).$$

Rearranging terms and applying known results from differential equations completes the proof. □

Lemma 4 (Bids below values). *Suppose marginal values v are continuous. In any equilibrium bids are weakly below marginal values for all achievable quantities q, $G^i(q; b) < 1$.*

Proof. Let $\hat{b} = b \wedge v$, and suppose that $b > \hat{b}$ on some set of achievable quantities. Submitting the bid function $\hat{b}$ instead of b yields a weakly lower expected payment even at allocations unaffected by the altered bid (due to the probabilistic implementation of discriminatory transfers). Because this implies that the deviation is weakly utility-improving for allocations unaffected by the deviation, it is sufficient to show that utility is also improved for all allocations directly affected by the deviation.

Suppose that aggregate supply Q leads to the allocation q_i, and that $b(q_i) > v(q_i) = \hat{b}(q_i)$. Because market clearing prices weakly decrease when bids decrease it follows that $p(Q) \geq \hat{p}(Q)$, the market clearing price when bidder i submits the bid function $\hat{b}$. Since $\hat{b}(q_i) = v(q_i) = p(Q) \geq \hat{p}(Q)$, when she submits the alternate bid function bidder i receives at least as much quantity as where her marginal value equaled the previous market clearing price, $\hat{q}_i \geq v^{-1}(p(Q)) \equiv \check{q}_i$. Expected utility, over the randomization of the payment rule, conditional on this aggregate quantity realization is

$$\underbrace{\int_0^{q_i} v\left(x\right) - \alpha b\left(x\right) dx - \left(1-\alpha\right) q_i b\left(q_i\right)}_{\text{utility under } b} \gtrless \underbrace{\int_0^{\hat{q}_i} v\left(x\right) - \alpha \hat{b}\left(x\right) dx - \left(1-\alpha\right) \hat{q}_i \hat{b}\left(\hat{q}_i\right)}_{\text{utility under } \hat{b}}.$$

Note that the left-hand side can be written as

$$\begin{aligned}
&\int_0^{q_i} v(x) - \alpha b(x)\, dx - (1-\alpha)\, q_i b(q_i) \\
&= \int_0^{\hat{q}_i} v(x) - \alpha b(x) - (1-\alpha)\, b(q_i)\, dx + \int_{\hat{q}_i}^{q_i} v(x) - \alpha b(x) - (1-\alpha)\, b(q_i)\, dx \\
&\leq \int_0^{\hat{q}_i} v(x) - \alpha \hat{b}(x) - (1-\alpha)\, \hat{b}(\hat{q}_i)\, dx + \int_{\hat{q}_i}^{q_i} v(x) - b(q_i)\,.
\end{aligned}$$

By assumption, $v(x) < b(q_i)$ for all $x \in (\hat{q}_i, q_i)$. Then as long as $\hat{q}_i < q_i$ this implies the deviation $\hat{b}$ is profitable. The allocation is unaffected by the deviation (conditional on $b(q_i) > v(q_i)$) only if b is discontinuous at q_i. Since b is monotone decreasing and bounded it is continuous almost everywhere, and thus since v is continuous there is some $q_i' < q_i$ at which b is continuous and $b(q_i') > v(q_i')$. This point will satisfy the conditions sufficient for the deviation to be profitable. □

Lemma 5 (No flat bids)**.** *Suppose that marginal values v are strictly decreasing. In any symmetric equilibrium bids are strictly decreasing at all achievable quantities q, $G^i(q;b) < 1$.*

Proof. Intuitively if there are flats in a symmetric bid function then tiebreaking must take place in equilibrium. Since bids are weakly below values, some agent has an incentive to break the tie in their favor with a slight upward deviation.

Suppose the bid function b is flat from $\underline{q}$ to $\overline{q}$, $b(q) = \overline{b}$ for all $q \in (\underline{q}, \overline{q})$; assume that this interval is maximal in the sense that $b(q') > b(q) > b(q'')$ for all $q' < \underline{q}$, $q'' > \overline{q}$, and $q \in (\underline{q}, \overline{q})$. Because bids are symmetric, quantities in this range will arise when the aggregate quantity is between $n\underline{q}$ and $n\overline{q}$.

Let $\varepsilon > 0$ and consider a deviation b^ε,

$$b^\varepsilon(q) = \begin{cases} b(q) & \text{if } b(q) \notin \left[\overline{b}, \overline{b} + \varepsilon\right], \\ \overline{b} + \varepsilon & \text{otherwise.} \end{cases}$$

For ε sufficiently small there is $\underline{q}_\varepsilon = \inf\{q : \; b^\varepsilon(q) > b(q)\}$ which is the minimum quantity affected by the deviation b^ε. Under this deviation, whenever $Q \in (n\underline{q}_\varepsilon, n\overline{q})$ bidder i's allocation is $q^{i,\varepsilon}(Q) = \min\{Q - (n-1)\underline{q}_\varepsilon, \overline{q}\}$. I now show that the expected utility from the deviation is greater than the expected utility from the original bid function. Eliminating portions of the expected utility function which remain unchanged this is

$$\int_{n\underline{q}_\varepsilon}^{(n-1)\underline{q}_\varepsilon+\overline{q}}\int_0^{Q-(n-1)\underline{q}_\varepsilon} v(x)-\alpha b^\varepsilon(x)\,dx-(1-\alpha)\left(Q-(n-1)\underline{q}_\varepsilon\right)\left(\overline{b}+\varepsilon\right)dF(Q)$$
$$+\int_{(n-1)\underline{q}_\varepsilon+\overline{q}}^{n\overline{q}}\int_0^{\overline{q}} v(x)-\alpha b^\varepsilon(x)\,dx-(1-\alpha)\overline{q}b\left(\frac{Q-\overline{q}}{n-1}\right)dF(Q)$$
$$-\alpha(1-F(n\overline{q}))\int_{\underline{q}_\varepsilon}^{\overline{q}} b^\varepsilon(x)-b(x)\,dx$$
$$>\int_{n\underline{q}_\varepsilon}^{n\overline{q}}\int_0^{q^i(Q)} v(x)-\alpha b(x)\,dx-(1-\alpha)q^i(Q)\,b\left(q^i(Q)\right)dF(Q).$$

Rearranging the inequality, we want to show that

$$\int_{n\underline{q}_\varepsilon}^{(n-1)\underline{q}_\varepsilon+\overline{q}}\int_{q^i(Q)}^{Q-(n-1)\underline{q}_\varepsilon} v(x)\,dxdF(Q)+\int_{(n-1)\underline{q}_\varepsilon+\overline{q}}^{n\overline{q}}\int_{q^i(Q)}^{\overline{q}} v(x)\,dxdF(Q)$$
$$>\int_{n\underline{q}_\varepsilon}^{(n-1)\underline{q}_\varepsilon+\overline{q}}\int_0^{Q-(n-1)\underline{q}_\varepsilon}\alpha b^\varepsilon(x)\,dx+(1-\alpha)\left(Q-(n-1)\underline{q}_\varepsilon\right)\left(\overline{b}+\varepsilon\right)dF(Q)$$
$$+\int_{(n-1)\underline{q}_\varepsilon+\overline{q}}^{n\overline{q}}\int_0^{\overline{q}}\alpha b^\varepsilon(x)\,dx+(1-\alpha)\overline{q}b\left(\frac{Q-\overline{q}}{n-1}\right)dF(Q)+\alpha(1-F(n\overline{q}))\int_{\underline{q}_\varepsilon}^{\overline{q}} b^\varepsilon(x)-b(x)\,dx$$
$$-\int_{n\underline{q}_\varepsilon}^{n\overline{q}}\int_0^{q^i(Q)}\alpha b(x)\,dx+(1-\alpha)q^i(Q)\,b\left(q^i(Q)\right)dF(Q).$$

As $\varepsilon\searrow 0$, $\underline{q}_\varepsilon\nearrow\underline{q}$ and $b^\varepsilon\searrow b$. Thus in the limit this inequality is

$$\int_{n\underline{q}}^{(n-1)\underline{q}+\overline{q}}\int_{q^i(Q)}^{Q-(n-1)\underline{q}} v(x)\,dxdF(Q)+\int_{(n-1)\underline{q}+\overline{q}}^{\overline{q}}\int_{q^i(Q)}^{\overline{q}} v(x)\,dxdF(Q)$$
$$>\int_{n\underline{q}}^{(n-1)\underline{q}+\overline{q}}\int_0^{Q-(n-1)\underline{q}}\alpha b(x)\,dx+(1-\alpha)\left(Q-(n-1)\underline{q}\right)\overline{b}dF(Q)$$
$$\int_{(n-1)\underline{q}+\overline{q}}^{n\overline{q}}\int_0^{\overline{q}}\alpha b(x)\,dx+(1-\alpha)\overline{q}\overline{b}dF(Q)$$
$$-\int_{n\underline{q}}^{n\overline{q}}\int_0^{q^i(Q)}\alpha b(x)\,dx+(1-\alpha)q^i(Q)\,\overline{b}dF(Q).$$

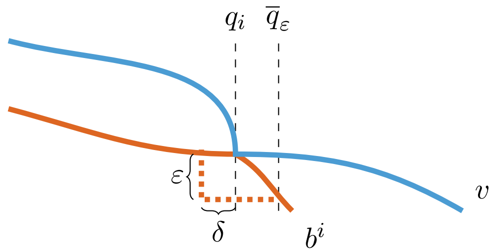

Figure 6: Construction of the deviation $b^{\varepsilon\delta}$ used to prove Lemma 6.

This can be consolidated to

$$\begin{aligned}
&\int_{n\underline{q}}^{n\overline{q}} \int_{q^i(Q)}^{\min\left\{Q-(n-1)\underline{q},\overline{q}\right\}} v\left(x\right) dx dF\left(Q\right) \\
&> \int_{n\underline{q}}^{n\overline{q}} \int_{q^i(Q)}^{\min\left\{Q-(n-1)\underline{q},\overline{q}\right\}} \alpha b\left(x\right) dx + \left(1-\alpha\right)\left(\min\left\{Q-\left(n-1\right)\underline{q},\overline{q}\right\} - q^i\left(Q\right)\right)\overline{b} dF\left(Q\right) \\
&= \int_{n\underline{q}}^{n\overline{q}} \int_{q^i(Q)}^{\min\left\{Q-(n-1)\underline{q},\overline{q}\right\}} b\left(x\right) dx dF\left(Q\right).
\end{aligned}$$

By Lemma 4 and the assumption that marginal values are strictly decreasing, $b(q) < v(q)$ for all $q \in (\underline{q}, \overline{q})$. Then so long as there is a bidder i such that $\Pr(q_i < \overline{q} | Q \in (n\underline{q}, n\overline{q})) > 0$ some agent will fit a deviation of the form b^ε utility-improving. Such a bidder must exist, as otherwise $\Pr(\sum_{i=1}^n q_i = n\overline{q} | Q \in (n\underline{q}, n\overline{q})) = 1$, violating market clearing. □

Lemma 6 (Bids are strictly below values)**.** *Suppose that marginal values are strictly decreasing and continuous. Then if $\alpha > 0$, $b^i(q) < v^i(q)$ for all q such that $G^i(q) < 1$.*

Proof. Suppose that there is q_i such that $b^i(q_i) = v^i(q_i)$. Let $\varepsilon, \delta > 0$ and consider a deviation $b^{\varepsilon\delta}$ given by

$$b^{\varepsilon\delta}\left(q\right) = \begin{cases} b^i\left(q\right) & \text{if } q < q_i - \delta, \\ b^i\left(q\right) & \text{if } b^i\left(q\right) < v^i\left(q_i\right) - \varepsilon, \\ v^i\left(q_i\right) - \varepsilon & \text{otherwise.} \end{cases}$$

By construction the deviation b^ε saves payment of at least $\alpha\varepsilon\delta$ with probability $1 - G^i(q_i)$.

The costs of the deviation b^ε can be divided into two categories. When under the original bid function b^i bidder i's allocation q is between $q_i - \delta$ and q_i, under the deviation it is at least $q_i - \delta$. Then the quantity lost is at most δ, and the margin per unit is at most $v(q_i - \delta) - v(q_i)$. Since v is continuous for any $\varepsilon > 0$ there is $\delta > 0$ such that $v(q_i - \delta) - v(q_i) < \varepsilon$. Taking appropriate δ the utility loss is then bounded by $\varepsilon\delta$, and occurs with probability $G^i(q_i; b_i) - G^i(q_i - \delta; b_i)$. Since this probability is going to zero in ε, this order-$\varepsilon\delta$ loss is dominated by the order-$\varepsilon\delta$ gain demonstrated above.

Let $\overline{q}_\varepsilon = \varphi^i(b^i(q_i + \varepsilon))$ be the quantity at which the deviation $b^{\varepsilon\delta}$ again equals b^i. When under the original bid function bidder i's allocation q is between q_i and $\overline{q}_\varepsilon$, under the deviation $b^{\varepsilon\delta}$ her allocation is at least $q_i - \delta$. In this interval the deviation yields lost utility of at most

$$\int_{q_i}^{\overline{q}_\varepsilon} \int_{q_i-\delta}^{q} v(x) - b^i(x)\, dx dG^i\left(q; b^i\right) \leq \int_{q_i}^{\overline{q}_\varepsilon} \int_{q_i}^{q} v(x) - b^i(x)\, dx + \varepsilon\delta dG^i\left(q; b^i\right).$$

The deviation yields cost savings for larger quantities, given by

$$\int_{q_i}^{\overline{q}_\varepsilon} b^i(x) - \left(b^i(q_i) - \varepsilon\right) dx \left(1 - G^i\left(\overline{q}_\varepsilon\right)\right) = \left(\left(\overline{q}_\varepsilon - q_i\right)\varepsilon + \int_{q_i}^{\overline{q}_\varepsilon} b^i(x) - b^i(q_i)\, dx\right)\left(1 - G^i\left(\overline{q}_\varepsilon; b^i\right)\right).$$

In the limit we can take δ to be arbitrarily small, so to show that deviation is profitable it is sufficient to show that for small $\varepsilon > 0$,

$$\left(\left(\overline{q}_\varepsilon - q_i\right)\varepsilon + \int_{q_i}^{\overline{q}_\varepsilon} b^i(x) - b^i(q_i)\, dx\right)\left(1 - G^i\left(\overline{q}_\varepsilon; b^i\right)\right) > \int_{q_i}^{\overline{q}_\varepsilon} \int_{q_i}^{q} v(x) - b^i(x)\, dx dG^i\left(q; b^i\right).$$

At $\varepsilon = 0$ the two sides both equal 0. Taking the derivative gives

$$\left(\left(\overline{q}_\varepsilon - q_i\right) + \frac{d\overline{q}_\varepsilon}{d\varepsilon}\varepsilon\right)\left(1 - G^i\left(\overline{q}_\varepsilon; b^i\right)\right) > \left(\left(\overline{q}_\varepsilon - q_i\right)\varepsilon + \int_{q_i}^{\overline{q}_\varepsilon} v(x) - b^i(q_i)\, dx\right) dG^i\left(\overline{q}_\varepsilon; b^i\right)\frac{d\overline{q}_\varepsilon}{d\varepsilon}$$
$$\geq \left(\overline{q}_\varepsilon - q_i\right)\varepsilon dG^i\left(\overline{q}_\varepsilon; b^i\right)\frac{d\overline{q}_\varepsilon}{d\varepsilon}.$$

Dividing both sides by ε transforms the desired inequality to

$$\left(\frac{\overline{q}_\varepsilon - q_i}{\varepsilon} + \frac{d\overline{q}_\varepsilon}{d\varepsilon}\right)\left(1 - G^i\left(\overline{q}_\varepsilon; b^i\right)\right) > \left(\overline{q}_\varepsilon - q_i\right) dG^i\left(\overline{q}_\varepsilon; b^i\right)\frac{d\overline{q}_\varepsilon}{d\varepsilon}.$$

Since v is strictly decreasing, Lemma 4 implies that $\lim_{\varepsilon\searrow 0}\overline{q}_\varepsilon = q_i$. By assumption, $\lim_{\varepsilon\searrow 0} G^i(\overline{q}_\varepsilon; b^i) < 1$. Then taking the limit of the above inequality, we have that our desired inequality is satisfied so long as $d\overline{q}_\varepsilon/d\varepsilon|_{\varepsilon=0} > 0$. If this derivative is zero but b^i is continuous, the derivative is strictly positive for ε near zero, and again the inequality is satisfied. If b^i is discontinuous, for ε sufficiently small the deviation $b^{\varepsilon\delta}$ yields no quantity loss (when the original bid function would have resulted in a quantity allocation $q > q_i$; that is, $\overline{q}_\varepsilon = q_i$) and gains exactly equal losses in this case. Since gains outweigh losses in the case in which $q \in (q_i - \delta, q_i]$ the deviation $b^{\varepsilon\delta}$ is profitable. □

Lemma 7 (Continuous bids have bounded slope)**.** *Let $\alpha > 0$ and suppose that marginal values are continuous. If b^i is differentiable at $q_i < q^i(\overline{Q})$, $b^i_q > -\infty$.*

Proof. Suppose that b^i_q is infinitely negative at q_i, and suppose that b^j is right-differentiable at $q_j = \varphi^j(b^i(q))$ for all $j \neq i$ such that b^j is right-continuous at q_j. Let $\varepsilon > 0$ and consider a deviation $\hat{b}^\varepsilon$ such that

$$\hat{b}^\varepsilon(q) = \begin{cases} b^i(q) & \text{if } q \notin [q_i, q_i + \varepsilon], \\ b^i(q_i + \varepsilon) & \text{otherwise.} \end{cases}$$

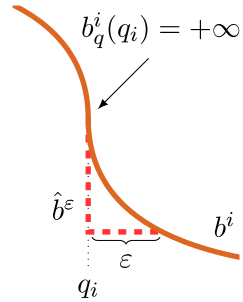

Figure 7: The deviation used to prove Lemma 7.

When the realized aggregate quantity Q is such that $q^i(Q) > q_i + \varepsilon$, the deviation b^ε saves payment

$$\alpha \int_{q_i}^{q_i+\varepsilon} b^i(q) - b^i(q_i+\varepsilon)\, dx \left(1 - G^i\left(q_i+\varepsilon; b^i\right)\right).$$

When the realized aggregate quantity Q is such that $q^i(Q) \in [q_i, q_i+\varepsilon]$ the deviation yields quantity that is bounded below by q_i (this follows from Lemma 5), hence the costs of the deviation are bounded above by

$$\int_{q_i}^{q_i+\varepsilon} \int_{q_i}^{q} v(x) - \alpha b^i(x)\, dx - (1-\alpha)(q-q_i)\, b^i(q)\, dG^i\left(q; b^i\right).$$

We now show that for small ε the benefits of the deviation outweigh the costs,

$$\begin{aligned} &\alpha \int_{q_i}^{q_i+\varepsilon} b^i(x) - b^i(q_i+\varepsilon)\, dx \left(1 - G^i\left(q_i+\varepsilon; b^i\right)\right) \\ &> \int_{q_i}^{q_i+\varepsilon} \int_{q_i}^{q} v(x) - \alpha b^i(x)\, dx - (1-\alpha)(q-q_i)\, b^i(q)\, dG^i\left(q; b^i\right). \end{aligned}$$

At $\varepsilon = 0$ both sides are zero, so we consider the first derivative of each. We want to show

$$\begin{aligned} &-\alpha b^i_q(q_i+\varepsilon)\left(1 - G^i\left(q_i+\varepsilon; b^i\right)\right) - \alpha \int_{q_i}^{q_i+\varepsilon} b^i(x) - b^i(q_i+\varepsilon)\, dx G^i_q\left(q_i+\varepsilon; b^i\right) \\ &> \left(\int_{q_i}^{q_i+\varepsilon} v(x) - \alpha b^i(x)\, dx - (1-\alpha)\varepsilon b^i(q_i+\varepsilon)\right) dG^i\left(q_i+\varepsilon; b^i\right). \end{aligned}$$

Rearranging gives a desired inequality of

$$\begin{aligned} &-\alpha b^i_q(q_i+\varepsilon)\left(1 - G^i\left(q_i+\varepsilon; b^i\right)\right) \\ &> \left(\int_{q_i}^{q_i+\varepsilon} v(x) - \alpha b^i(q_i+\varepsilon)\, dx - (1-\alpha)\varepsilon b^i(q_i+\varepsilon)\right) dG^i\left(q_i+\varepsilon; b^i\right). \end{aligned}$$

By assumption the left-hand side is unboundedly positive at $\varepsilon \searrow 0$. The left-hand multiplicand of the right-hand side is bounded by $\bar{\nu}\varepsilon$ so it is sufficient to show that $dG^i(q_i+\varepsilon;b)$ is at most proportional to b^i_q as $\varepsilon \searrow 0$.

By definition,

$$G^i\left(q;b^i\right) = F\left(q + \sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right)$$
$$\implies \quad dG^i\left(q;b^i\right) = f\left(q + \sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right)\left(1 + \sum_{j\neq i}\varphi^j_p\left(b^i\left(q\right)\right)b^i_q\left(q\right)\right).$$

Then as long as φ^j_p is finite at $b^i(q)$ for all $j \neq i$ — that is, opponent bids are not locally flat at q_j — the proof is complete. From the first-order conditions of the model,

$$-\left(v^j\left(q\right) - b^j\left(q\right)\right)\left(n-1\right)\sum_{k\neq j}\varphi^k_p\left(b^j\left(q\right)\right)f\left(Q\right) = \alpha\left(1 - F\left(Q\right)\right) + \left(1-\alpha\right)qf\left(q\right),$$
$$Q = q + \sum_{k\neq j}\varphi^k\left(b^j\left(q\right)\right).$$

Since all exogenous terms are bounded, if φ^j_p is infinite at q_j it must be that $v^i(q_i) = b^i(q_i)$. This contradicts Lemma 6. □

Lemma 8 (Marginal quantity response is continuous). *Let $\alpha > 0$. For all agents i and almost all p achieveable in equilibrium, $\sum_{j\neq i}\varphi^j_p$ is continuous.*

Proof. Bidder i's first-order conditions are given by

$$-\left(v^j\left(q\right) - b^j\left(q\right)\right)\left(n-1\right)\sum_{k\neq j}\varphi^k_p\left(b^j\left(q\right)\right)f\left(Q\right) = \alpha\left(1 - F\left(Q\right)\right) + \left(1-\alpha\right)qf\left(q\right),$$
$$Q = q + \sum_{k\neq j}\varphi^k\left(b^j\left(q\right)\right).$$

Then if $\sum_{j\neq i}\varphi^j_p(p)$ is discontinuous, bidder i's ideal best response bid function is discontinuous. If her ideal bid function has an upward discontinuity the bid monotonicity constraint must be binding on some interval. On this interval, $\varphi^i_p = \infty$, hence her opponents' bids must equal their values. This contradicts Lemma 6.

If bidder i''s ideal bid function has a downward discontinuity it must be the case that $\sum_{j\neq i}\varphi^j_p(p)$ has an upward discontinuity (the margin per unit has an upward discontinuity, so the slope of opponent demand has a magnitude-decreasing discontinuity). At this point of discontinuity, $\varphi^i_p(p) = 0$. Then for all $j \neq i$ except potentially one, $\sum_{k\neq j}\varphi^k_p(p)$ has an upward discontinuity, and all but one of i's opponents have a downward discontinuity in their idealized bid functions at the inverse demand associated price p. Then there is an interval $(\underline{p}, \overline{p}) \ni p$ such that no bidder submits a bid in this interval.

Let bidder j be such that there is q_j with $\lim_{q \nearrow q_j} b^j(q) = \overline{p}$. Let $\varepsilon > 0$ and construct a deviation b^ε,

$$b^\varepsilon(q) = \begin{cases} b^j(q) & \text{if } q \notin (q_j - \varepsilon, q_j], \\ \underline{p} & \text{otherwise.} \end{cases}$$

This deviation saves payment of at least $\alpha\varepsilon(\overline{p} - \underline{p})$ with probability $1 - G^i(q_j; b^j)$, which is greater than zero by assumption. It sacrifices utility of at most $(v(q_j - \varepsilon) - \overline{p})\varepsilon$ with probability $G^i(q_j; \varepsilon) - G^i(q_j - \varepsilon; \varepsilon)$. This probability is going to zero by assumption that bidder j is among the "final" bidders above $\overline{p}$, so the deviation b^ε is utility-improving. It follows that $\sum_{j \neq i} \varphi_p^j$ is continuous. □

Theorem 11 (Equilibrium is symmetric ($\alpha > 0$))**.** *If marginal values are strictly decreasing and continuous, and $\alpha > 0$, then all equilibria are symmetric.*

Proof. Assume that bids are right-continuous at 0. There are at least two bidders submitting the highest possible bid, $b^i(0) = b^j(0) = \overline{b}$, $\overline{b} = \max_k b^k(0)$, otherwise the lone high bidder has a strict incentive to reduce her bid.

Consider the set of bidders k for whom $b^k(0) = \overline{b}$. It cannot be the case that two bidders submit initially flat bids, and all bidders whose bids are initially decreasing submit bids with the same initial slope (otherwise there is variation in $\sum_{j \neq k} \varphi_p^k(\overline{b})$, implying different upper bids). Suppose that bidder i is such that $b^i(0) = \overline{b}$, and her bid is constant until some quantity $\check{q} > 0$. Lemma 7 implies that her opponents who also submit the maximum possible bid are submitting bids which are Lipschitz continuous at 0; let the m be the modulus of continuity. If bidder j is such that $b^j(0) = \overline{b}$, let $\varepsilon > 0$ and construct a deviation b^ε by

$$b^\varepsilon(q) = \begin{cases} \overline{b} + \varepsilon & \text{if } q < \varepsilon, \\ b^j(q) & \text{otherwise.} \end{cases}$$

When ε is sufficiently small this deviation incurs additional costs which are bounded by $(m+1)\varepsilon^2$, with probability one. It also results in additional utility of at least $\int_0^\varepsilon v(x) - (\overline{b} + \varepsilon)dx$ with probability bounded below by $F(\check{q}) - F(\varepsilon)$.[24] Lemma 4 implies that since $b^i(0) = \overline{b}$ and b^i is flat until $\check{q}$, $v(x) - \overline{b} = \mu > 0$. Then for ε small the utility gain is bounded below by $(\mu/2 + \varepsilon)\varepsilon$. Since $\lim_{\varepsilon \searrow 0} F(\varepsilon) = 0$, the gain from deviation is order-ε which the associated costs are order-ε^2, implying that deviation is profitable. Then no agent can submit an initially flat bid in equilibrium. It follows that all bidders with $b^j(0) = \overline{b}$ submit bids which initially have the same slope.

Standard resuls from differential equations are sufficient to show that bid functions are identical for all bidders who submit the same initial bid $b^i(0)$. Then if equilibrium is asymmetric it must be that there are different classes of bidders, determined by initial bid. Assume that equilibrium is asymmetric and let $\overline{b}'$ be the next-highest initial bid, $\overline{b}' = \max\{b^k(0) : b^k(0) \neq \overline{b}\}$. Let bidder i be such that $b^i(0) = \overline{b}'$. By Lemma 7 it must be that $b_q^i(0)$ is finite, and hence $\varphi_p^i(0) > 0$. Lemma 8 implies that $\sum_{k \neq j} \varphi_p^k$ is continuous at $\overline{b}'$, so the symmetric-response-within-class argument implies

[24] These bounds can be easily improved, but the argument is simpler from these naïve bounds.

that all bidders j for whom $b^j(0) = \overline{b}$ have an upward kink in φ^j at $\overline{b}'$. Since this is the case, a slight upward deviation by bidder i at quantity $q = 0$ results in a strict increase in her first-order condition, implying that this upward shift is utility-improving (entering at $\overline{b}'$ fails the second-order condition). This contradicts bidder i entering at a lower price. □

Theorem 12 (Equilibrium is symmetric ($\alpha = 0$))**.** *If marginal values are decreasing and continuous, and $\alpha = 0$, then all equilibria are symmetric.*

Proof. This proof is essentially identical to that in Klemperer and Meyer (1989).

We first show that all bidders have identical initial bids, and initial bid slopes. When $\alpha = 0$, the bidder's first-order condition for optimality, (8), is

$$-\left(v\left(q\right) - b^i\left(q\right)\right)\sum_{j\neq i}\varphi_p^j\left(b^j\left(q\right)\right) = q.$$

Then when $q = 0$, $b^i(0) = v(0)$. Rearranging gives

$$\lim_{q\searrow 0}\frac{v\left(q\right) - b^i\left(q\right)}{q} = v_q\left(0\right) - b_q^i\left(0\right) = -\frac{1}{\sum_{j\neq i}\varphi_p^j\left(v\left(0\right)\right)}. \tag{9}$$

Suppose that bidders i and j are such that $b^i \neq b^j$. Let $R \equiv \sum_{k\neq i,j}\varphi_p^k(v(0))$. Subtracting bidder j's (9) from bidder i's (9) gives

$$\begin{aligned} b_q^j\left(0\right) - b_q^i\left(0\right) &= \frac{1}{R+\varphi_p^i\left(v\left(0\right)\right)} - \frac{1}{R+\varphi_p^j\left(v\left(0\right)\right)} \\ \implies &\left(1+\frac{1}{Rb_q^j\left(0\right)+1}\right)b_q^j\left(0\right) = \left(1+\frac{1}{Rb_q^i\left(0\right)+1}\right)b_q^i\left(0\right). \end{aligned} \tag{10}$$

This is trivially solved when $b_q^i(0) = b_q^j(0)$. Furthermore,

$$\begin{aligned} \frac{d}{db_q^j\left(0\right)}\left[\left(1+\frac{1}{Rb_q^j\left(0\right)+1}\right)b_q^j\left(0\right)\right] &= \left(1+\frac{1}{Rb_q^j\left(0\right)+1}\right) - \frac{Rb_q^j\left(0\right)}{\left(Rb_q^j\left(0\right)+1\right)^2} \\ &\overset{\text{sign}}{=} \left(Rb_q^j\left(0\right)+2\right)\left(Rb_q^j\left(0\right)+1\right) - Rb_q^j\left(0\right) \\ &= \left(Rb_q^j\left(0\right)+1\right)^2 + 1 > 0. \end{aligned}$$

Then the left-hand side of (10) is strictly monotone in $b_q^j(0)$, and there is a unique solution to (10). Basic algebra then gives $(n-2)b_q(0) = (n-1)v_q(0)$.[25]

Now, suppose there are bidders i and j for whom $\varphi^i \neq \varphi^j$. Without loss of generality, assume $\varphi^i(p) > \varphi^j(p)$, and let $\underline{p} \equiv \inf\{p' : \varphi^i(p'') > \varphi^j(p'')\ \forall p'' \in (p', p)\}$. Since $\varphi^i(\bar{b}) = \varphi^j(\bar{b})$ for

[25]The fundamental theorem of ordinary differential equations does not apply here, since the differential system is not Lipschitz continuous. In particular, when specified in the appropriate form $b' = f(q, b)$, the slope of f goes to infinity as $q \searrow 0$. This is only the case when $\alpha = 0$; when $\alpha > 0$, the value of $b(0)$ fully determines the entire bid curve.

$\bar{b} = b^i(0) = b^j(0)$, for $\tilde{p} > \underline{p}$ but sufficiently close to $\underline{p}$, it must be that $\varphi^i(\tilde{p}) > \varphi^j(\tilde{p})$ and $\varphi^i_p(\tilde{p}) > \varphi^j_p(\tilde{p})$. Following the first order conditions for optimality (when $\alpha = 0$), we have

$$\begin{aligned}\left(v\left(\varphi^i\left(\tilde{p}\right)\right) - \tilde{p}\right) &= -\frac{\varphi^i\left(\tilde{p}\right)}{\varphi^j_p\left(\tilde{p}\right) + \sum_{k\neq i,j}\varphi^k_p\left(\tilde{p}\right)} \\ &> -\frac{\varphi^j\left(\tilde{p}\right)}{\varphi^i_p\left(\tilde{p}\right) + \sum_{k\neq i,j}\varphi^k_p\left(\tilde{p}\right)} = \left(v\left(\varphi^j\left(\tilde{p}\right)\right) - \tilde{p}\right) > \left(v\left(\varphi^i\left(\tilde{p}\right)\right) - \tilde{p}\right).\end{aligned}$$

This is a contradiction, hence there cannot exist an asymmetric equilibrium. □

B Equilibrium uniqueness

Lemma 9 (Equilibrium necessary condition)**.** *A necessary condition for a market price function p to represent an equilibrium is*

$$\frac{1-2\alpha}{1-\alpha}\left(p\left(\overline{Q}\right) - \hat{v}\left(\overline{Q}\right)\right) + \overline{Q}\hat{v}_Q\left(\overline{Q}\right) \leq 0.$$

Proof. Suppose that $p(\overline{Q}) < v(\overline{Q}/n)$. Then for sufficiently small $\varepsilon > 0$, there is $\delta > 0$ such that for all $q \in (\overline{Q}/n - \delta, \overline{Q}/n)$ it must be that

$$v\left(\frac{1}{n}\overline{Q}\right) > \lim_{q' \nearrow \frac{1}{n}Q} b^i\left(q'\right) + \varepsilon > b^i\left(q\right) \geq \lim_{q' \nearrow \frac{1}{n}Q} b^i\left(q'\right).$$

Define the limiting price $\overline{b} = \lim_{q' \nearrow \overline{Q}/n} b^i(q')$, and let $\overline{\varepsilon} = v(\overline{Q}/n) - \overline{b}$. For $\varepsilon \in (0, \overline{\varepsilon})$, define $q_\varepsilon = \inf\{q :\ b^i(q) \leq \overline{b} + \varepsilon\}$. Define a deviation $\hat{b}_\varepsilon$ such that

$$\hat{b}_\varepsilon\left(q\right) = \begin{cases} b^i\left(q\right) & \text{if } q < q_\varepsilon, \\ \overline{b} + \varepsilon & \text{if } q \geq q_\varepsilon. \end{cases}$$

Deviating to $\hat{b}_\varepsilon$ affects outcomes only when the realization of supply $Q > nq_\varepsilon$. In this case, the bidder receives all additional quantity $Q > nq_\varepsilon$, and also pays more for units which would have been won anyway, Q/n. The costs of the deviation must outweigh the benefits, hence

$$\begin{aligned}\Delta u \equiv \int_{nq_\varepsilon}^{\overline{Q}} &\underbrace{\int_{\frac{1}{n}Q}^{Q-(n-1)q_\varepsilon} v\left(y\right) - \hat{b}_\varepsilon\left(y\right) dy}_{\text{add'l quantity}} - \underbrace{\alpha \int_{q_\varepsilon}^{\frac{1}{n}Q} \hat{b}_\varepsilon\left(y\right) - b\left(y\right) dy}_{\text{add'l discriminatory payment}} \\ &- \underbrace{\left(1-\alpha\right)\left(\frac{1}{n}Q\right)\left(\hat{b}_\varepsilon\left(\frac{1}{n}Q\right) - b\left(\frac{1}{n}Q\right)\right)}_{\text{add'l uniform price payment}} dF\left(Q\right) \leq 0.\end{aligned}$$

To simplify analysis, I rearrange this expression to be

$$\Delta u = \int_{nq_\varepsilon}^{\overline{Q}} \int_{\frac{1}{n}Q}^{Q-(n-1)q_\varepsilon} v(y)\,dy - Qb(q_\varepsilon) + (n-(1-\alpha))\,q_\varepsilon b(q_\varepsilon) + \alpha \int_{q_\varepsilon}^{\frac{1}{n}Q} b(y)\,dy + (1-\alpha)\frac{1}{n}Qb\left(\frac{1}{n}Q\right) dF(Q) \leq 0.$$

When $\varepsilon = 0$, $nq_\varepsilon = \overline{Q}$ and $\Delta u = 0$. I show below that the same is true of both the first and second derivatives. I therefore derive a condition on the third derivative of Δu: when this derivative is negative, Δu will be negative for small $\varepsilon > 0$, a necessary condition for equilibrium.

To simplify analysis, let $b_\varepsilon \equiv \overline{b} + \varepsilon = b(q_\varepsilon)$. The first derivative of Δu is

$$\begin{aligned}\frac{d\Delta u}{d\varepsilon} &= ndq_\varepsilon \left[\int_{q_\varepsilon}^{q_\varepsilon} v(y)\,dy - nq_\varepsilon b_\varepsilon + (n-(1-\alpha))\,q_\varepsilon b_\varepsilon + \alpha\int_{q_\varepsilon}^{q_\varepsilon} b(y)\,dy + (1-\alpha)\,q_\varepsilon b_\varepsilon\right] f(nq_\varepsilon) \\ &\quad + \int_{nq_\varepsilon}^{\overline{Q}} -(n-1)\,dq_\varepsilon v(Q-(n-1)\,q_\varepsilon) - Q \\ &\qquad + (n-(1-\alpha))\,dq_\varepsilon b_\varepsilon + (n-(1-\alpha))\,q_\varepsilon - \alpha dq_\varepsilon b_\varepsilon dF(Q) \\ &= \int_{nq_\varepsilon}^{\overline{Q}} -(n-1)\,(v(Q-(n-1)\,q_\varepsilon) - b_\varepsilon)\,dq_\varepsilon - Q + (n-(1-\alpha))\,q_\varepsilon dF(Q).\end{aligned}$$

Since $nq_\varepsilon|_{\varepsilon=0} = \overline{Q}$, it follows that $d\Delta u/d\varepsilon|_{\varepsilon=0} = 0$.

The second derivative of Δu is

$$\begin{aligned}\frac{d^2\Delta u}{d\varepsilon^2} &= -ndq_\varepsilon\left[-(n-1)\,(v(q_\varepsilon) - b_\varepsilon)\,dq_\varepsilon - nq_\varepsilon + (n-(1-\alpha))\,q_\varepsilon\right] f(nq_\varepsilon) \\ &\quad + \int_{nq_\varepsilon}^{\overline{Q}} -(n-1)\,(-(n-1)\,v_q(Q-(n-1)\,q_\varepsilon)\,dq_\varepsilon - 1)\,dq_\varepsilon \\ &\qquad - (n-1)\,(v(Q-(n-1)\,q_\varepsilon) - b_\varepsilon)\,d^2q_\varepsilon + (n-(1-\alpha))\,dq_\varepsilon dF(Q) \\ &= -ndq_\varepsilon\left[-(n-1)\,(v(q_\varepsilon) - b_\varepsilon)\,dq_\varepsilon - (1-\alpha)\,q_\varepsilon\right] f(nq_\varepsilon) \\ &\quad + \int_{nq_\varepsilon}^{\overline{Q}} -(n-1)\,(-(n-1)\,v_q(Q-(n-1)\,q_\varepsilon)\,dq_\varepsilon - 1)\,dq_\varepsilon \\ &\qquad - (n-1)\,(v(Q-(n-1)\,q_\varepsilon) - b_\varepsilon)\,d^2q_\varepsilon + (n-(1-\alpha))\,dq_\varepsilon dF(Q).\end{aligned}$$

As in the case of the first derivative, the integral term in the second derivative drops out when $\varepsilon = 0$. The leading additive term is proportional to

$$-(n-1)\left(v\left(\frac{1}{n}\overline{Q}\right) - b\left(\frac{1}{n}\overline{Q}\right)\right)\frac{q}{b_q\left(\frac{1}{n}\overline{Q}\right)} - (1-\alpha)\frac{1}{n}\overline{Q}.$$

This is exactly the first-order condition at $q = \overline{Q}/n$, and is therefore equal to zero. Then $d^2\Delta u/d\varepsilon^2|_{\varepsilon=0} = 0$.

To avoid unnecessary complications, when taking the third derivative I omit the inner integral derivative, since previous arguments imply that when $\varepsilon = 0$, the integral will evaluate to zero. Since the derivative can be nontrivially signed without this integral term, it can be ignored. Denote by $\tilde{d}^3\Delta u/\tilde{d}\varepsilon^3$ the third derivative of Δu, without this integral term.

$$\begin{aligned}
\frac{\tilde{d}^3\Delta u}{\tilde{d}\varepsilon^3} &= -nd^2q_\varepsilon \underbrace{\left[-(n-1)(v(q_\varepsilon)-b_\varepsilon)dq_\varepsilon-(1-\alpha)q_\varepsilon\right]}_{\text{FOC}=0|_{\varepsilon=0}} f(nq_\varepsilon) \\
&\quad - n^2dq_\varepsilon^2 \underbrace{\left[-(n-1)(v(q_\varepsilon)-b_\varepsilon)dq_\varepsilon-(1-\alpha)q_\varepsilon\right]}_{\text{FOC}=0|_{\varepsilon=0}} df(nq_\varepsilon) \\
&\quad - ndq_\varepsilon\left[-(n-1)(v_q(q_\varepsilon)dq_\varepsilon-1)dq_\varepsilon-(n-1)(v(q_\varepsilon)-b_\varepsilon)d^2q_\varepsilon-(1-\alpha)dq_\varepsilon\right]f(nq_\varepsilon) \\
&\quad - ndq_\varepsilon\left[\begin{array}{l} -(n-1)(-(n-1)v_q(q_\varepsilon)dq_\varepsilon-1)dq_\varepsilon \\ \qquad -(n-1)(v(q_\varepsilon)-b_\varepsilon)d^2q_\varepsilon+(n-(1-\alpha))dq_\varepsilon \end{array}\right]f(nq_\varepsilon).
\end{aligned}$$

Noting that $-dq_\varepsilon \geq 0$, when evaluated at $\varepsilon = 0$ the (adjusted) third derivative is proportional to

$$\begin{aligned}
\left.\frac{\tilde{d}^3\Delta u}{\tilde{d}\varepsilon^3}\right|_{\varepsilon=0} &\propto -(n-1)(v_q(q_\varepsilon)dq_\varepsilon-1)dq_\varepsilon-(n-1)(v(q_\varepsilon)-b_\varepsilon)d^2q_\varepsilon-(1-\alpha)dq_\varepsilon \\
&\quad +(n-1)^2v_q(q_\varepsilon)dq_\varepsilon^2+(n-1)dq_\varepsilon+(n-1)(v(q_\varepsilon)-b_\varepsilon)d^2q_\varepsilon+(n-(1-\alpha))dq_\varepsilon \\
&= (n-1)(n-2)v_q(q_\varepsilon)dq_\varepsilon^2-2(n-1)(v(q_\varepsilon)-b_\varepsilon)d^2q_\varepsilon \\
&\quad + \underbrace{\left[2(n-1)-(1-\alpha)+(n-(1-\alpha))\right]}_{3n+2\alpha-4}dq_\varepsilon.
\end{aligned}$$

Implicit differentiation gives $dq_\varepsilon = 1/b_q$, and $d^2q_\varepsilon = -b_{qq}/b_q^3$. Then, letting $Q^\mu = \overline{Q}/n$, the (adjusted) third derivative is weakly negative if and only if

$$(n-1)(n-2)v_q(Q^\mu)+2(n-1)(v(Q^\mu)-b(Q^\mu))\frac{b_{qq}(Q^\mu)}{b_q(Q^\mu)}+(3n+2\alpha-4)b_q(Q^\mu) \leq 0.$$

In equilibrium,

$$
\begin{aligned}
-\left(n-1\right)\left(v\left(q\right)-b\left(q\right)\right)\frac{f\left(nq\right)}{b_q\left(q\right)} &= \alpha\left(1-F\left(nq\right)\right)+\left(1-\alpha\right)qf\left(nq\right) \\
\implies -\left(n-1\right)\left(v_q\left(Q^\mu\right)-b_q\left(Q^\mu\right)\right)\frac{f\left(\overline{Q}\right)}{b_q\left(Q^\mu\right)} & \\
-n\left(n-1\right)\left(v\left(Q^\mu\right)-b\left(Q^\mu\right)\right)\frac{df\left(\overline{Q}\right)}{b_q\left(Q^\mu\right)} & \\
+\left(n-1\right)\left(v\left(Q^\mu\right)-b\left(Q^\mu\right)\right)\frac{f\left(\overline{Q}\right)b_{qq}\left(Q^\mu\right)}{b_q\left(Q^\mu\right)^2} &= -n\alpha f\left(\overline{Q}\right)+\left(1-\alpha\right)f\left(\overline{Q}\right)+n\left(1-\alpha\right)\overline{Q}df\left(\overline{Q}\right) \\
\implies -\left(n-1\right)\left(v_q\left(Q^\mu\right)-b_q\left(Q^\mu\right)\right) & \\
+\left(n-1\right)\left(v\left(Q^\mu\right)-b\left(Q^\mu\right)\right)\frac{b_{qq}\left(Q^\mu\right)}{b_q\left(Q^\mu\right)} &= \left(1-\left(n+1\right)\alpha\right)b_q\left(Q^\mu\right).
\end{aligned}
$$

Substituting this into the desired inequality gives

$$
\begin{aligned}
&\left(n-1\right)\left(n-2\right)v_q\left(Q^\mu\right)+2\left[\left(n-1\right)\left(v_q\left(Q^\mu\right)-b_q\left(Q^\mu\right)\right)+\left(1-\left(n+1\right)\alpha\right)b_q\left(Q^\mu\right)\right] \\
&\quad+\left(3n+2\alpha-4\right)b_q\left(Q^\mu\right) \\
&= n\left(n-1\right)v_q\left(Q^\mu\right)+\left(-2\left(n-1\right)+2\left(1-\left(n+1\right)\alpha\right)+\left(3n+2\alpha-4\right)\right)b_q\left(Q^\mu\right) \\
&= n\left(n-1\right)v_q\left(Q^\mu\right)+\left(n-2n\alpha\right)b_q\left(Q^\mu\right) \\
&= n\left(n-1\right)v_q\left(Q^\mu\right)-\left(n-2n\alpha\right)\left(n-1\right)\left(v\left(Q^\mu\right)-b\left(Q^\mu\right)\right)\frac{1}{\left(1-\alpha\right)Q^\mu} \\
&\propto Q^\mu v_q\left(Q^\mu\right)-\frac{1-2\alpha}{1-\alpha}\left(v\left(Q^\mu\right)-b\left(Q^\mu\right)\right)\leq 0.
\end{aligned}
$$

Evaluated at $\varepsilon=0$, the third derivative is

$$
\begin{aligned}
-\Bigg[&-2\left(n-1\right)\left(\hat{v}\left(\overline{Q}\right)-\overline{b}\right)\frac{d^2q_\varepsilon}{d\varepsilon}+\left(n-1\right)\left(n-2\right)n\hat{v}_Q\left(\overline{Q}\right)\left(\frac{dq_\varepsilon}{d\varepsilon}\right)^2 \\
&+\left(2\alpha+3n-4\right)\frac{dq_\varepsilon}{d\varepsilon}\Bigg]n\frac{dq_\varepsilon}{d\varepsilon}dF\left(\overline{Q}\right)\leq 0.
\end{aligned}
$$

Noting that $dq_\varepsilon/d\varepsilon=1/np_Q$, $d^2q_\varepsilon/d\varepsilon^2=-p_{QQ}/np_Q^3$, and substituting in for the solution to the agent's first-order conditions, this is equivalent to

$$
-2\left(n-1\right)\frac{p_{QQ}\left(\overline{Q}\right)}{\tilde{H}\left(\overline{Q}\right)}+\left(n-1\right)\left(n-2\right)\hat{v}_Q\left(\overline{Q}\right)+\left(2\alpha+3n-4\right)p_Q\left(\overline{Q}\right)\leq 0.
$$

Since $p_Q = (p - \hat{v})\tilde{H}$, substitution into this expression gives

$$-2(n-1)\left(p\left(\overline{Q}\right)-\hat{v}\left(\overline{Q}\right)\right)\left(\tilde{H}\left(\overline{Q}\right)+\frac{\tilde{H}_Q\left(\overline{Q}\right)}{\tilde{H}\left(\overline{Q}\right)}\right)+2(n-1)\hat{v}_Q\left(\overline{Q}\right)$$
$$+(n-1)(n-2)\hat{v}_Q\left(\overline{Q}\right)+(2\alpha+3n-4)\left(p\left(\overline{Q}\right)-\hat{v}\left(\overline{Q}\right)\right)\tilde{H}\left(\overline{Q}\right)\leq 0.$$

Finally, replacing $\tilde{H}$ and $\tilde{H}_Q$ yields

$$\frac{1}{(1-\alpha)\overline{Q}}(n-2\alpha n)\left(p\left(\overline{Q}\right)-\hat{v}\left(\overline{Q}\right)\right)+n\hat{v}_Q\left(\overline{Q}\right)\leq 0.$$

The desired inequality is immediate. □

Lemma 10 (α-monotonicity of necessary condition)**.** *Suppose that p is a solution to the equilibrium market-price equation for randomization α, and satisfies the necessary condition of Lemma 2. Then for any $\alpha' < \alpha$, there is a solution to the equilibrium market-price equation p' with $p'(\overline{Q}) = p(\overline{Q})$ which satisfies the necessary condition of Lemma 2.*

Proof. Note that the existence of a solution is simply a claim that there is a C' that provides $p'(\overline{Q}) = p(\overline{Q})$. This is trivial. It is necessary then only to establish the latter claim in the Lemma.

To recapitulate, the necessary condition of Lemma 2 is

$$\frac{1-2\alpha}{1-\alpha}\left(p\left(\overline{Q}\right)-\hat{v}\left(\overline{Q}\right)\right)+\overline{Q}\hat{v}_Q\left(\overline{Q}\right)\leq 0.$$

If this inequality is increasing in α, the claim will be established. Notice that the right-hand term is constant in α while, by assumption, $p(\overline{Q}) - \hat{v}(\overline{Q})$ is constant and negative. Hence it is sufficient to show that

$$\frac{d}{d\alpha}\left[\frac{1-2\alpha}{1-\alpha}\right]\leq 0.$$

Basic algebra reveals the derivative to be

$$\frac{-2}{1-\alpha}+\frac{1-2\alpha}{(1-\alpha)^2}=-\left(\frac{1}{1-\alpha}\right)^2<0.$$

Thus if p satisfies the endpoint condition of Lemma 2 for randomization parameter α, p' satisfies the endpoint condition for randomization parameter $\alpha' < \alpha$. □

C Polynomial-Lomax model

C.1 Conjugate equilibrium

Proof of Theorem 2. In equilibrium, market prices solve

$$p_Q(Q) = (p(Q) - \hat{v}(Q))\tilde{H}(Q). \tag{11}$$

In the polynomial-Lomax model

$$\tilde{H}(Q) = \frac{(n-1)\lambda}{n\alpha\overline{Q} + ((1-\alpha)\lambda - n\alpha) Q} \equiv \frac{c_1}{c_2 + c_3 Q}.$$

Then equation (11) can be written

$$(c_2 + c_3 Q) p_Q(Q) = (p(Q) - \hat{v}(Q)) c_1. \tag{12}$$

When bids are conjugate, the market price is a polynomial of the same order as marginal value. Matching coefficients gives a system of equations,

$$\begin{aligned} \overline{k} c_3 p_{\overline{k}} &= \left(p_{\overline{k}} - \hat{v}_{\overline{k}}\right) c_1 & \implies \quad p_{\overline{k}} &= \frac{c_1 \hat{v}_{\overline{k}}}{c_1 - \overline{k} c_3}; \\ (k+1) c_2 p_{k+1} + k c_3 p_k &= (p_k - \hat{v}_k) c_1 \quad \left(k < \overline{k}\right) & \implies \quad p_k &= \frac{c_1 \hat{v}_k + (k+1) c_2 p_{k+1}}{c_1 - k c_3}. \end{aligned}$$

Solving this system of equations gives

$$p_k = c_1 \sum_{t=k}^{\bar{k}} \left(\frac{t!}{k!}\right) \left(\prod_{s=k}^{t} \frac{1}{c_1 - s c_3}\right) c_2^{t-k} \hat{v}_t = \frac{c_1}{c_2} \sum_{t=k}^{\bar{k}} \left(\frac{t!}{k!}\right) \left(\prod_{s=k}^{t} \frac{c_2}{c_1 - s c_3}\right) \hat{v}_t.$$[26]

Substituting in for c_1, c_2, c_3, and $\hat{v}$ gives the result. □

Lemma 11. *When* $(\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3} x^k \neq 1/x$,

$$\int_0^Q (\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3} x^k dx = \frac{1}{(\hat{c}_3 + 1)\hat{c}_2} \left((\hat{c}_1 + \hat{c}_2 Q)^{\hat{c}_3 + 1} Q^k - k \int_0^Q (\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3 + 1} x^{k-1} dx \right).$$

Proof. This is obtained by direct computation.

$$\begin{aligned} &\int_0^Q (\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3} x^k dx \\ &= \frac{1}{\hat{c}_2}\left(\frac{1}{\hat{c}_3 + 1}\right) \left[(\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3 + 1} x^k\right]\Big|_{x=0}^{Q} - \frac{k}{\hat{c}_2}\left(\frac{1}{\hat{c}_3 + 1}\right) \int_0^Q (\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3 + 1} x^{k-1} dx \\ &= \frac{1}{(\hat{c}_3 + 1)\hat{c}_2} \left((\hat{c}_1 + \hat{c}_2 Q)^{\hat{c}_3 + 1} Q^k - k \int_0^Q (\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3 + 1} x^{k-1} dx \right). \end{aligned}$$

□

[26] It is tempting to simplify this expression by bringing the factorial into the product, but this requires the assumption that $0/0 = 1$ in the case of $k = 0$. Since $0! = 1$ this is not necessary is the slightly more unwieldy formulation.

Lemma 12. *When $k \in \mathbb{N}_+$ and $(\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3} x^k \neq 1/x$,*

$$\int_0^Q (\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3} x^k dx = \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i \left(\prod_{j=0}^{i} \frac{k+1-j}{(\hat{c}_3 + (j+1))\,\hat{c}_2} \right) (\hat{c}_1 + \hat{c}_2 Q)^{\hat{c}_3 + (i+1)} Q^{k-i}.$$

Proof. This is recursion on Lemma 11.

When $k = 0$, Lemma 11 gives

$$\int_0^Q (\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3} dx = \frac{1}{(\hat{c}_3 + 1)\,\hat{c}_2} (\hat{c}_1 + \hat{c}_2 Q)^{\hat{c}_3 + 1}.$$

This validates the desired result in the case of $k = 0$.

When $k > 0$, Lemma 11 and the recursive hypothesis give

$$\begin{aligned}
&\int_0^Q (\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3} x^k dx \\
&= \frac{1}{(\hat{c}_3 + 1)\,\hat{c}_2} \left((\hat{c}_1 + \hat{c}_2 Q)^{\hat{c}_3 + 1} Q^k - k \int_0^Q (\hat{c}_1 + \hat{c}_2 x)^{\hat{c}_3 + 1} x^{k-1} dx \right) \\
&= \frac{1}{(\hat{c}_3 + 1)\,\hat{c}_2} (\hat{c}_1 + \hat{c}_2 Q)^{\hat{c}_3 + 1} Q^k \\
&\quad - \frac{k}{(\hat{c}_3 + 1)\,\hat{c}_2} \left[\frac{1}{k} \sum_{i=0}^{k-1} (-1)^i \left(\prod_{j=0}^{i} \frac{k-j}{(\hat{c}_3 + (j+2))\,\hat{c}_2} \right) (\hat{c}_1 + \hat{c}_2 Q)^{\hat{c}_3 + (i+2)} Q^{(k-1)-i} \right] \\
&= \frac{1}{(\hat{c}_3 + 1)\,\hat{c}_2} (\hat{c}_1 + \hat{c}_2 Q)^{\hat{c}_3 + 1} Q^k \\
&\quad + \frac{1}{k+1} \sum_{i=1}^{k} (-1)^i \left(\prod_{j=0}^{i} \frac{k+1-j}{(\hat{c}_3 + (j+1))\,\hat{c}_2} \right) (\hat{c}_1 + \hat{c}_2 Q)^{\hat{c}_3 + (i+1)} Q^{k-i} \\
&= \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i \left(\prod_{j=0}^{i} \frac{k+1-j}{(\hat{c}_3 + (j+1))\,\hat{c}_2} \right) (\hat{c}_1 + \hat{c}_2 Q)^{\hat{c}_3 + (i+1)} Q^{k-i}.
\end{aligned}$$

This validates the desired result in the case of $k > 0$. □

Proof of Theorem 3. A direct implication of Lemma 5 is that the bid monotonicity constraint is never strictly binding, and hence can be ignored. In equilibrium bids must satisfy the first-order conditions pointwise almost everywhere. To establish the desired result it is then sufficient to evaluate equation (5).

In the polynomial-Lomax model we have

$$\tilde{H}(Q) = \frac{(n-1)\,\lambda}{n\alpha\left(\overline{Q} - Q\right) + (1-\alpha)\,\lambda Q} \equiv \frac{c_1}{c_2 + c_3 Q},$$

$$\exp\left(\int_0^Q \tilde{H}(x)\,dx \right) = \left(\frac{c_2 + c_3 Q}{c_2} \right)^{\frac{c_1}{c_3}}.$$

Then to establish the desired result it is sufficient to prove that there are $\tilde{p}_k$, $k \in \{0, \ldots, \bar{k}\}$, such that

$$\int_0^Q \exp\left(-\int_0^x \tilde{H}(y)\,dy\right) \tilde{H}(x)\,\hat{v}(x)\,dx \exp\left(\int_0^Q \tilde{H}(x)\,dx\right) = \sum_{k=0}^{\bar{k}} \tilde{p}_k Q^k. \tag{13}$$

We then evaluate

$$\int_0^Q \left(\frac{c_2 + c_3 Q}{c_2}\right)^{-\frac{c_1}{c_3}} \left(\frac{c_1}{c_2 + c_3 Q}\right) \sum_{k=0}^{\bar{k}} \hat{v}_k x^k dx = c_1 c_2^{\frac{c_1}{c_3}} \sum_{k=0}^{\bar{k}} \hat{v}_k \int_0^Q (c_2 + c_3 Q)^{-\frac{c_1 + c_3}{c_3}} x^k dx.$$

Lemma 12 implies that there are constants $\gamma_{k,t}$ such that this can be written as

$$c_1 c_2^{\frac{c_1}{c_3}} \sum_{k=0}^{\bar{k}} \hat{v}_k \int_0^Q (c_2 + c_3 Q)^{-\frac{c_1 + c_3}{c_3}} x^k dx = c_1 c_2^{\frac{c_1}{c_3}} \sum_{k=0}^{\bar{k}} \hat{v}_k \sum_{t=0}^{k} \gamma_{k,t} \left(1 + \frac{c_3}{c_2} Q\right)^{t - \frac{c_1}{c_3}} Q^{k-t}.$$

The left-hand side of equation (13) is postmultiplied by $\exp(\int_0^Q \tilde{H}(x)dx) = (1 + c_3 Q/c_2)^{c_1/c_3}$. Then we have

$$\begin{aligned} & c_1 c_2^{\frac{c_1}{c_3}} \sum_{k=0}^{\bar{k}} \hat{v}_k \int_0^Q (c_2 + c_3 Q)^{-\frac{c_1 + c_3}{c_3}} x^k dx \exp\left(\int_0^Q \tilde{H}(x)\,dx\right) \\ &= c_1 c_2^{\frac{c_1}{c_3}} \sum_{k=0}^{\bar{k}} \hat{v}_k \sum_{t=0}^{k} \gamma_{k,t} \left(1 + \frac{c_3}{c_2} Q\right)^{t} Q^{k-t}. \end{aligned}$$

Simple expansion shows that this is an order-$\bar{k}$ polynomial in Q. Then all solutions to the market clearing equation (5) are an order-$\bar{k}$ polynomial plus a residual nonpolynomial term. Standard arguments from differential equations imply that this polynomial is equal to the conjugate market price function, which Theorem 2 establishes to be a solution to the symmetric equilibrium first order conditions. □

C.2 Equilibrium revenue

Lemma 13. *The moments of the Lomax distribution are given by*

$$\mathbb{E}\left[Q^k\right] = \prod_{t=1}^{k} \frac{t\overline{Q}}{\lambda + t}.$$

Proof. When $k > 0$ direct computation yields

$$\begin{aligned}\mathbb{E}\left[Q^k\right] &= \frac{\lambda}{\overline{Q}}\int_0^{\overline{Q}}\left(1-\frac{Q}{\overline{Q}}\right)^{\lambda-1}Q^k dQ \\ &= -\left(1-\frac{Q}{\overline{Q}}\right)^{\lambda}Q^k\Bigg|_{Q=0}^{\overline{Q}} + k\int_0^{\overline{Q}}\left(1-\frac{Q}{\overline{Q}}\right)^{\lambda}Q^{k-1}dQ \\ &= k\int_0^{\overline{Q}}\left(1-\frac{Q}{\overline{Q}}\right)^{\lambda}Q^{k-1}dQ = \frac{k\overline{Q}}{\lambda}\mathbb{E}\left[Q^{k-1}\right] - \frac{k}{\lambda}\mathbb{E}\left[Q^k\right].\end{aligned}$$

It follows that

$$\mathbb{E}\left[Q^k\right] = \frac{k\overline{Q}}{\lambda+k}\mathbb{E}\left[Q^{k-1}\right].$$

Since $\mathbb{E}[Q^0] = 1$, the result follows. □

Proof of Theorem 9. In the linear-Lomax model price coefficients are

$$p_0 = v_0 + \frac{1}{n}\left(\frac{n\alpha\overline{Q}}{(n-2)\lambda + (n+\lambda)\alpha}\right)v_1, \quad p_1 = \frac{1}{n}\left(\frac{(n-1)\lambda}{(n-2)\lambda+(n+\lambda)\alpha}\right)v_1$$

We can expand the expected revenue formula in the linear-Lomax case,

$$\mathbb{E}[\pi] = p_0\mathbb{E}\left[Q^1\right] + \left(1-\frac{1}{2}\alpha\right)p_1\mathbb{E}\left[Q^2\right] = \frac{1}{(\lambda+1)(\lambda+2)}\left((\lambda+2)p_0\overline{Q} + (2-\alpha)p_1\overline{Q}^2\right).$$

The derivatives of the price coefficients are

$$\begin{aligned}\frac{\partial p_1}{\partial\alpha} &= -\frac{n+\lambda}{n}(n-1)\left((n-2)\lambda+(n+\lambda)\alpha\right)^{-2}\lambda v_1 \\ &= -\left(\frac{(n+\lambda)n}{(n-1)\lambda v_1}\right)p_1^2; \\ \frac{\partial p_0}{\partial\alpha} &= \frac{1}{n}\left(\frac{\left((n-2)\lambda+(n+\lambda)\alpha\right)n\overline{Q} - (n+\lambda)n\alpha\overline{Q}}{\left((n-2)\lambda+(n+\lambda)\alpha\right)^2}\right)v_1 \\ &= \left(\frac{(n-2)\lambda n^2\overline{Q}}{(n-1)^2\lambda^2 v_1}\right)p_1^2.\end{aligned}$$

Then the derivative of expected revenue is (proportional to)

$$\begin{aligned}\frac{\partial}{\partial\alpha}\mathbb{E}\left[\pi\right] &\propto \left(\lambda+2\right)\frac{\partial p_0}{\partial\alpha} - p_1\overline{Q} + \left(2-\alpha\right)\frac{\partial p_1}{\partial\alpha}\overline{Q} \\ &\propto \left(\lambda+2\right)\left(n-2\right)n\lambda\overline{Q}v_1 - \left(n-1\right)\left(\left(n-2\right)\lambda + \left(n+\lambda\right)\alpha\right)\lambda\overline{Q}v_1 \\ &\quad - \left(2-\alpha\right)\left(n+\lambda\right)\left(n-1\right)\lambda\overline{Q}v_1 \\ &\bar{\propto} \left(\lambda+2\right)\left(n-2\right)n - \left(n-1\right)\left(\left(n-2\right)\lambda + \left(n+\lambda\right)\alpha\right) - \left(2-\alpha\right)\left(n+\lambda\right)\left(n-1\right) \\ &= \left(\lambda+2\right)\left(n-2\right)n - \left(n-1\right)\left(n-2\right)\lambda - 2\left(n+\lambda\right)\left(n-1\right) \\ &= \left(\lambda+2\right)\left(n-2\right)n - \left(n-1\right)\lambda n - 2\left(n-1\right)n \\ &\propto \left(\lambda+2\right)\left(n-2\right) - \left(\lambda+2\right)\left(n-1\right) \\ &= -\left(2+\lambda\right).\end{aligned}$$

Replacing the terms eliminated by division gives

$$\frac{\partial}{\partial\alpha}\mathbb{E}\left[\pi\right] = -\frac{\lambda}{\lambda+1}\left(\frac{1}{\left(n-2\right)\lambda+\left(n+\lambda\right)\alpha}\right)^2\overline{Q}^2 v_1 > 0.$$

Then the original derivative is strictly positive, and expected revenue is increasing in α. □

Proof of Theorem 10. In general, the conjugate equilibrium of the uniform price auction will not be revenue-maximizing. Then to compare discriminatory auction revenue against uniform price auction revenue, I analyze non-conjugate equilibrium by setting the homogeneous term so that $b(\overline{Q}/n) = v(\overline{Q}/n)$. Per the market clearing equation, this is

$$\begin{aligned}\hat{v}\left(\overline{Q}\right) &= \left(\overline{C} - \int_0^{\overline{Q}}\exp\left(-\int_0^x \tilde{H}\left(y\right)dy\right)\tilde{H}\left(x\right)\hat{v}\left(x\right)dx\right)\exp\left(\int_0^{\overline{Q}}\tilde{H}\left(x\right)dx\right) \\ \implies \overline{C} &= \hat{v}\left(\overline{Q}\right)\exp\left(-\int_0^{\overline{Q}}\tilde{H}\left(x\right)dx\right) + \int_0^{\overline{Q}}\exp\left(-\int_0^x\tilde{H}\left(y\right)dy\right)\tilde{H}\left(x\right)dx\hat{v}\left(x\right)dx.\end{aligned}$$

Then equilibrium market clearing prices are

$$\begin{aligned}\overline{p}\left(Q\right) &= \hat{v}\left(\overline{Q}\right)\exp\left(-\int_Q^{\overline{Q}}\tilde{H}\left(x\right)dx\right) + \int_Q^{\overline{Q}}\exp\left(-\int_Q^x\tilde{H}\left(y\right)dy\right)\tilde{H}\left(x\right)\hat{v}\left(x\right)dx \\ &= \hat{v}\left(Q\right) + \int_Q^{\overline{Q}}\exp\left(-\int_Q^x\tilde{H}\left(y\right)dy\right)\hat{v}_Q\left(x\right)dx.\end{aligned}$$

The latter equality follows from integration by parts.

Now, note that

$$\tilde{H}^{\mathrm{PAB}}(x) = \frac{n-1}{n}\left(\frac{f(x)}{1-F(x)}\right) \quad \Longrightarrow \quad \exp\left(-\int_Q^x \tilde{H}^{\mathrm{PAB}}(y)\,dy\right) = \left(\frac{1-F(x)}{1-F(Q)}\right)^{\frac{n-1}{n}},$$
$$\tilde{H}^{\mathrm{UPA}}(x) = \frac{n-1}{x} \quad \Longrightarrow \quad \exp\left(-\int_Q^x \tilde{H}^{\mathrm{UPA}}(y)\,dy\right) = \left(\frac{Q}{x}\right)^{n-1}.$$

Substituting in for the Lomax distribution, this gives

$$\overline{p}^{\mathrm{PAB}}(Q) = \hat{v}(Q) + \int_Q^{\overline{Q}} \left(\frac{\overline{Q}-x}{\overline{Q}-Q}\right)^{\left(\frac{n-1}{n}\right)\lambda} \hat{v}_Q(x)\,dx,$$
$$\overline{p}^{\mathrm{UPA}}(Q) = \hat{v}(Q) + \int_Q^{\overline{Q}} \left(\frac{Q}{x}\right)^{n-1} \hat{v}_Q(x)\,dx.$$

In the linear-Lomax model, $\hat{v}(Q) = \hat{v}_0 + \hat{v}_1 Q$, where $\hat{v}_0 > 0$ and $\hat{v}_1 < 0$. Substituting in then gives a closed form for revenue-maximizing equilibrium prices,

$$\begin{aligned}
\overline{p}^{\mathrm{PAB}}(Q) &= \hat{v}_0 + \hat{v}_1 Q + \frac{n}{n+(n-1)\lambda}\hat{v}_1\left(\overline{Q}-Q\right) \\
&= \left[\hat{v}_0 + \frac{n\hat{v}_1\overline{Q}}{n+(n-1)\lambda}\right] + \left[\frac{(n-1)\lambda}{n+(n-1)\lambda}\right]\hat{v}_1 Q, \\
\overline{p}^{\mathrm{UPA}}(Q) &= \hat{v}_0 + \hat{v}_1 Q + \frac{1}{2-n}Q^{n-1}\left(\overline{Q}^{2-n} - Q^{2-n}\right)\hat{v}_1 \\
&= \hat{v}_0 + \left[\frac{1-n}{2-n}\right]\hat{v}_1 Q + \left[\frac{\overline{Q}^{2-n}}{2-n}\right]\hat{v}_1 Q^{n-1}.
\end{aligned}$$

In each equilibrium, expected revenue is given by

$$\mathbb{E}\left[\pi^{\mathrm{PAB}}\right] = \int_0^{\overline{Q}} (1-F(Q))\,\overline{p}^{\mathrm{PAB}}(Q)\,dQ,$$
$$\mathbb{E}\left[\pi^{\mathrm{UPA}}\right] = \int_0^{\overline{Q}} Q\overline{p}^{\mathrm{UPA}}(Q)\,f(Q)\,dQ = \int_0^{\overline{Q}} (1-F(Q))\left(\overline{p}^{\mathrm{UPA}}(Q) + Q\overline{p}_Q^{\mathrm{UPA}}(Q)\right)dQ.$$

From the above, we have

$$\overline{p}^{\mathrm{UPA}}(Q) + Q\overline{p}^{\mathrm{UPA}}(Q) = \hat{v}_0 + 2\left[\frac{1-n}{2-n}\right]\hat{v}_1 Q + n\left[\frac{\overline{Q}^{2-n}}{2-n}\right]\hat{v}_1 Q^{n-1}.$$

Then the difference in expected revenue is

$$
\begin{aligned}
&\mathbb{E}\left[\pi^{\text{PAB}} - \pi^{\text{UPA}}\right] \\
&= \int_0^{\overline{Q}} (1 - F(Q)) \left(\left[\frac{n\hat{v}_1 \overline{Q}}{n + (n-1)\lambda} \right] + \left[\frac{(n-1)\lambda \hat{v}_1}{n + (n-1)\lambda} - \frac{2(1-n)\hat{v}_1}{2-n} \right] Q - n \left[\frac{\overline{Q}^{2-n}}{2-n} \right] \hat{v}_1 Q^{n-1} \right) dQ \\
&\propto \int_0^{\overline{Q}} \left(\overline{Q} - Q\right)^{\lambda} \Big(\left[n(2-n)\overline{Q} \right] + \left[(2-n)(n-1)\lambda - 2(1-n)(n + (n-1)\lambda) \right] Q \\
&\qquad\qquad - \left[(n + (n-1)\lambda) n \overline{Q}^{2-n} \right] Q^{n-1} \Big) dQ \\
&\propto \int_0^{\overline{Q}} \left(\overline{Q} - Q\right)^{\lambda} \left([2-n]\overline{Q} + [(n-1)(2+\lambda)] Q - \left[(n + (n-1)\lambda) \overline{Q}^{2-n} \right] Q^{n-1} \right) dQ \\
&= [2-n] \frac{\overline{Q}^{\lambda+2}}{\lambda+1} + [(n-1)(2+\lambda)] \frac{\overline{Q}^{\lambda+2}}{(\lambda+1)(\lambda+2)} - \left[(n + (n-1)\lambda) \overline{Q}^{2-n} \right] \frac{\overline{Q}^{\lambda+n}}{\lambda+1} \prod_{m=2}^{n} \frac{m-1}{\lambda+m} \\
&\propto [2-n] + [n-1] - \left[(n + (n-1)\lambda) \prod_{m=2}^{n} \frac{m-1}{\lambda+m} \right] = 1 - (n + (n-1)\lambda) \prod_{m=2}^{n} \frac{m-1}{\lambda+m}. \qquad (14)
\end{aligned}
$$

When $n = 3$, this is

$$
1 - (3 + 2\lambda) \frac{2}{(\lambda+2)(\lambda+3)} = 1 - \frac{6 + 4\lambda}{6 + 5\lambda + \lambda^2} > 0.
$$

Let $n' = n + 1$. Note that

$$
\begin{aligned}
\frac{(n' + (n'-1)\lambda) \prod_{m=2}^{n'} \frac{m-1}{\lambda+m}}{(n + (n-1)\lambda) \prod_{m=2}^{n'} \frac{m-1}{\lambda+m}} < 1 \qquad &\iff \qquad ((n+1) + n\lambda) n < (n + (n-1)\lambda)(\lambda + n + 1) \\
&\iff \qquad 0 < (n-1)(1+\lambda). \qquad \checkmark
\end{aligned}
$$

Then the product term in (14) is decreasing in n. Since the difference in revenue is strictly positive when $n = 3$, it is strictly positive for all n, and the discriminatory auction generates strictly more revenue than any equilibrium of the uniform price auction. $\square$