Auctions of Homogeneous Goods: A Case for Pay-as-Bid

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Abstract

Pay-as-bid is a prominent auction format for selling homogenous goods such as treasury securities and commodities. For the pay-as-bid auction we prove the uniqueness of pure-strategy Bayesian-Nash equilibria, establish a sufficient condition for the existence of this equilibrium, and obtain an unexpectedly tractable representation of equilibrium bids. Building on these results we analyze the optimal design of pay-as-bid auctions, as well as uniform-price auctions (the main alternative auction format). We show that seller's transparency about supply is optimal in pay-as-bid but not necessarily in uniform-price; pay-as-bid is weakly revenue dominant while the welfare comparison depends on the equilibrium selected in uniform-price; and, under a strategy selection commonly applied in empirical work, the two formats are revenue and welfare equivalent.

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1 Introduction

Each year, securities and commodities worth trillions of dollars are allocated through multi-unit auctions. The two primary auction formats for these sales are pay-as-bid and uniform-price. Pay-as-bid is the more popular of the two auction formats for selling treasury securities, and is frequently implemented to distribute electricity generation. It is also used in other government operations, including recent large-scale asset purchases in the U.S. (quantitative easing), and is implicitly run in financial markets when limit orders are followed by a market order.[1]

In this paper we analyze the design of pay-as-bid auctions. To accomplish this goal, we derive an equilibrium in bidding strategies, conditional on the auction's design, then we solve the auctioneer's problem, taking subsequent bidding equilibria as given. Our first contribution lies in expanding the bounds within which bidding behavior in the pay-as-bid auction is tractable: we allow an arbitrary number of bidders and general demands. In analyzing the bidding equilibrium we focus primarily on the case where bidders are symmetrically informed, and we leverage these results to obtain insights that are also valid in the presence of small informational asymmetries. Approximately symmetric information among bidders is a simplifying assumption which seems to be a good approximation in some important environments. For instance, any issue of treasury securities has both close substitutes whose prices are known, and the forward contracts based on the issue are traded ahead of the auction in the forward markets, thus providing bidders with substantial information about each others' valuations.[2] At the same time we allow for an arbitrary informational asymmetry between the seller and the bidders; we thus allow situations in which bidders' information was not available to the seller at the time the auction was designed, as well as situations in which the seller designs the auction format once and uses it for many auctions. We also allow for uncertainty of the total supply available for auction; exogenous supply uncertainty is a feature of some securities auctions.

Our theory of equilibrium bidding in pay-as-bid auctions focuses on pure-strategy equi-

[1]Pay-as-bid auctions are also referred to as discriminatory, or multiple-price auctions. Brenner et al. [2009] find that 33 of 48 countries surveyed allocate securities via pay-as-bid auction; Del Río [2017] finds that 27 of 31 markets surveyed distribute electricity generation via pay-as-bid auction (see also Maurer and Barroso [2011]). In both settings, most of the remaining markets are cleared by uniform-price auction. For financial markets, see, e.g., Glosten [1994].

[2]Hortaçsu et al. [2018] note that bids in U.S. Treasury auctions are typically "flat," which in light of our analysis supports the view that the bidders are close to being completely informed. Of course, our assumptions do not present a good approximation in some environments, in which there may be substantive asymmetries among bidders. Nonetheless, our assumptions are substantially milder than those typically employed by the rich prior literature on pay-as-bid auctions: with the exception of flat-demand environments, single-unit demand, and two-bidder-two-units examples, prior analyses of equilibrium bidding restrict attention to symmetrically-informed bidders with linear demands (see our discussion below).

libria. For the case with symmetric bidder information, we prove that equilibrium is unique and that bids have an unexpectedly tractable closed-form representation. We also establish a sufficient condition for the existence of equilibrium; our condition is satisfied e.g. when there are sufficiently many bidders. Going beyond symmetric information, we show that the seller's revenue in asymmetric-information equilibria of an optimally designed pay-as-bid auction is approximately bounded below by the revenue in the limit symmetric-information case.[3]

Our main design result establishes the revenue-optimality of transparently setting supply in pay-as-bid auctions: when bidders have symmetric information, revenue in the unique pure-strategy equilibrium is maximized when supply is deterministic; and when bidders have small informational asymmetries, revenue is approximately maximized when supply is deterministic. Thus determining the optimal supply distribution is equivalent to solving a standard monopoly problem. We show that limiting supply and limiting price in this problem are separable, and that at the optimum the seller does not need to consider how the limiting price affects whether bidders are constrained by the limiting quantity. Moreover, we show that the seller who cannot design the supply distribution wants to commit to reveal the realization of supply and that the seller's revenue is increasing in second-order spreads of the bidders' common information.[4] If the auctioneer is unable to influence the distribution of supply but still learns it before the auction is run, we show that it is optimal to commit to announce supply before bids are submitted. That is, it is optimal for the seller to inform bidders of the supply available, regardless of her ability to influence its distribution; this is in sharp contrast with the uniform-price auction, where (we show) deterministic supply is not necessarily revenue-optimal.[5]

[3]We obtain this bound even though it is not clear—nor do we resolve—whether in the limit, as asymmetric information vanishes, asymmetric-information equilibria converge to a pure-strategy equilibrium in the symmetric-information game. The subtlety is not only that different subsequences might in principle converge to different strategy profiles but also that a convergent subsequence might converge to a mixed-strategy profile that is different from the unique pure-strategy equilibrium.

[4]Because the seller in our model can set both a limiting quantity and limiting price, this monopoly problem is not entirely "standard." Nonetheless, it is straightforward to envision a monopolist setting both a limiting price and a limiting quantity. While in this discussion we focus on the seller setting reserve price and distribution of supply, in Appendix A we show that our insights also extend to the case when the seller can set a distribution over elastic supply curves; this extension relies on a Myerson-like regularity assumption imposed on bidders' values. In practice, in many treasury auctions the distribution of supply is partially determined by the demand from non-competitive bidders, and revenue maximization may not be auctioneer's only objective. However, treasuries and central banks have the ability to influence supply distributions, as well as to release data on non-competitive bids to competitive bidders; in this context our result provides a revenue-maximizing benchmark.

[5]The reason for this failure is the multiplicity of equilibria in uniform-price auctions. Although there is a uniform-price auction with deterministic supply admitting a revenue-optimal equilibrium, these auctions also admit zero-revenue equilibria (see, e.g., Burkett and Woodward [2020b] as well as other low revenue equilibria, see, e.g. LiCalzi and Pavan [2005], McAdams [2007], Kremer and Nyborg [2004], and Marszalec

We leverage our results on equilibrium bidding and supply transparency in the pay-as-bid auction to compare revenues and welfare in optimally-designed pay-as-bid and uniform-price auctions. We prove that the pay-as-bid format always raises weakly higher revenue, while the welfare comparison depends on equilibrium selection in uniform-price auction, which in general allows for multiple equilibria. Approximate revenue dominance and ambiguous welfare comparison remain valid under small asymmetries in bidder information. Major empirical studies comparing revenues between pay-as-bid and uniform-price auctions consider strategy profiles in which bidders in uniform-price bid truthfully for the marginal unit.[6] We show that truthful bidding is one equilibrium of an optimally-designed uniform-price auction and, under this equilibrium selection, we prove that both revenue and welfare are the same in pay-as-bid and uniform-price auction formats. Thus our results provide a theoretical explanation for the approximate revenue equivalence found by empirical work (see our discussion below).

Before situating our results in the rich related literature, we describe how the pay-as-bid auction operates. First, the bidders submit bids for each infinitesimal unit of the good. Then, the supply is realized, and the auctioneer (or, the seller) allocates the first infinitesimal unit to the bidder who submitted the highest bid, then the second infinitesimal unit to the bidder who submitted the second-highest bid, etc.[7] Each bidder pays her bid for each unit she obtains. The monotonic nature of how units are allocated implies that a collection of bids a bidder submitted can be equivalently described as a reported demand curve that is weakly-decreasing in quantity, but not necessarily continuous; the ultimate allocation resembles that of a classical Walrasian market, in which supply equals demand at a market-clearing price. We study pure-strategy Bayesian-Nash equilibria of this auction.[8]

We provide two sufficient conditions for equilibrium existence: a complex condition that is more general but difficult to analyze, and a simple condition that is less general but straight-

et al. [2020]). Depending on the auctioneer's concern about equilibrium selection, anticipated revenue may improve with some randomization (see our analysis of robust uniform-price bidding in Appendix I and Klemperer and Meyer [1989]).

[6]See e.g. Kang and Puller [2008], Hortaçsu and McAdams [2010], and Hortaçsu, Kastl, and Zhang [2018], and our discussion below.

[7]To fully-specify the auction we need to specify a tie-breaking rule; we adopt the standard tie-breaking rule, pro-rata on the margin, but our theory of equilibrium bidding does not hinge on this choice. This is in contrast to uniform-price auction, where tie-breaking matters; see Kremer and Nyborg [2004]. Additionally, in our transparency analysis, we consider the possibility that supply is realized before bids are submitted, but bidders remain unaware of its realization.

[8]In equilibrium, each bidder responds to the stochastic residual supply (that is, the supply given the bids of the remaining bidders). Effectively, the bidder is picking a point on each residual supply curve. In determining her best response, a bidder needs to keep in mind that: (i) the bid that is marginal if a particular residual supply curve is realized is paid not only when it is marginal, but also in any other state of nature that results in a larger allocation, and hence the bidder faces tradeoffs across these different states of nature; and (ii) bid curves need to be weakly monotonic in quantity, potentially a binding constraint.

forward. Our simple condition reduces the existence question to checking the optimization properties pointwise. It is satisfied, for instance, in the linear-Pareto settings analyzed by the prior literature discussed above, as well as for convex marginal values and for any distribution of supply provided there are sufficiently many bidders.[9] There is a large literature on equilibrium existence in pay-as-bid auctions. In symmetric-information settings, in addition to the contributions discussed above, Holmberg [2009] proves the existence of equilibrium when the distribution of supply has a decreasing hazard rate, and recognizes the possibility that (pure-strategy) equilibrium may not exist.[10] Our sufficient condition for existence encompasses the prior conditions and is substantially milder: with a sufficient number of bidders all distributions, including for instance the truncated normal distribution, satisfy our condition.[11] In asymmetric information settings, Athey [2001], McAdams [2003], and Reny [2011] have shown that equilibrium exists in multi-unit (discrete) pay-as-bid auctions, and Woodward [2019a] extends these results to the divisible-good context that we study.[12] A key difference between the results in these papers and ours is that the presence of private information allows the purification of mixed-strategy equilibria; such purification is not possible in the symmetric-information instances of our setting. Our existence conditions are consequences of our uniqueness and representation theorems, and (unlike general existence results) are not independent of the form of equilibrium.

Our theorems establishing the existence and uniqueness of pure-strategy Bayesian-Nash equilibrium in pay-as-bid auctions are reassuring for sellers using the pay-as-bid format; indeed, there are well-known problems posed by multiplicity of equilibria in other multi-unit auction formats.[13] Uniqueness is also important for the empirical study of pay-as-bid auctions. Estimation strategies based on the first-order conditions, or the Euler equation, rely on agents playing comparable equilibria across auctions in the data (Février et al. [2002], Hortaçsu and McAdams [2010], Hortaçsu and Kastl [2012], and Cassola et al. [2013]).[14] Equilibrium uniqueness plays an even larger role in the study of counterfactuals (see Armantier

[9]For many distributions of interest our condition is also satisfied with relatively few bidders; we provide examples in Section 3 and Appendix J.

[10]See also Fabra et al. [2006], Genc [2009], and Anderson et al. [2013] for discussions of potential problems with equilibrium existence.

[11]Prior literature conjectured that equilibrium cannot exist for truncated normal distributions.

[12]For equilibrium existence in multi-unit auctions, see also Břeský [1999], Jackson et al. [2002], Reny and Zamir [2004], Jackson and Swinkels [2005], Břeský [2008], and Kastl [2012]. Milgrom and Weber [1985] show existence of mixed-strategy equilibria.

[13]We establish the uniqueness of bids for relevant quantities—that is, for quantities a bidder wins with positive probability. Bids for quantities never obtained play no role in equilibrium outcomes. Our uniqueness result does not apply to these irrelevant bids.

[14]Maximum likelihood-based estimation strategies (e.g. Donald and Paarsch [1992]) also rely on agents playing comparable equilibria across auctions in the data. Chapman et al. [2005] discuss the requirement of comparability of data across auctions.

and Sbaï [2006]).[15]

Uniqueness was studied by Wang and Zender [2002] who prove the uniqueness of "nice" equilibria under strong parametric assumptions on utilities and distributions. Assuming that marginal values are linear and the supply is drawn from an unbounded Pareto distribution, they analyzed symmetric equilibria in which bids are piecewise continuously-differentiable functions of quantities and supply is invertible from equilibrium prices; they showed the uniqueness of such equilibria. Holmberg [2009] restricted attention to symmetric equilibria in which bid functions are twice differentiable, and—assuming that the maximum supply strictly exceeds the maximum total quantity the bidders are willing to buy—proved the uniqueness of such smooth and symmetric equilibria.[16] Ewerhart et al. [2010] and Ausubel et al. [2014] independently expand these analyses to Pareto supply with bounded support and linear marginal values. Restricting attention to equilibria in which bids are linear functions of quantities, they showed the uniqueness of such linear equilibria. In contrast, we look at all Bayesian-Nash equilibria of our model, we impose no parametric assumptions (not even continuity) and we do not require that some part of the supply is not wanted by any bidder.[17]

Our uniqueness result is also related to Klemperer and Meyer [1989] who established uniqueness in a duopoly model closely related to uniform-price auctions: when two symmetric and uninformed firms face random demand with unbounded support, then there is a unique equilibrium in their model.[18] The main difference between the two papers is, of course, that Klemperer and Meyer analyze the uniform-price format, while we look at pay-as-bid.[19] More generally, our uniqueness result provides a pay-as-bid counterpart for the literature on

[15] See also, in a related context, Cantillon and Pesendorfer [2006].

[16] Holmberg's assumption that bidders do not want to buy part of the supply represents a physical constraint in the reverse pay-as-bid electricity auction he studies: in his paper bidders supply electricity and face capacity constraints—beyond a certain level they cannot produce more. This low-capacity assumption drives the analysis and it precludes directly applying the same model in the context of securities auctions in which bidders are always willing to buy more (provided the price is sufficiently low).

[17] As a consequence of this generality, we need to develop a methodological approach which differs from that of the prior literature. McAdams [2002] and Ausubel et al. [2014] have also established the uniqueness of equilibrium in their respective parametric examples with two bidders and two goods.

[18] The analogue of their unbounded support assumption is our assumption that the support of supply extends all the way to no supply. While the two assumptions look analogous they have very different practical implications. In a treasury auction, for example, a seller can guarantee that with some tiny probability the supply will be lower than the target; in fact, in practice the supply is often random and our support assumption is satisfied. On the other hand, it is substantially more difficult, and practically impossible, for the seller to guarantee the risk of arbitrarily-large supplies. Note also that we have known since Wilson [1979] that the uniform-price auction may admit multiple equilibria. No similar multiplicity constructions exist for pay-as-bid auctions.

[19] The proof of our uniqueness result follows a differential analysis familiar from uniqueness results for first-price auctions (see, e.g., Maskin and Riley [2003], Lizzeri and Persico [2000], and Lebrun [2006]), but our analysis anchoring the differential machinery to a single reference point at which all equilibria must coincide is distinct.

collusive-seeming equilibria in uniform-price auctions. In particular, by proving equilibrium uniqueness for pay-as-bid we show its resilience to equilibrium collusion. For the analysis of collusive-seeming equilibria in uniform-price, in addition to Klemperer and Meyer [1989] who point out that the auctioneer can induce competition in a uniform-price auction by introducing slight randomness in supply, Kremer and Nyborg [2004] look at the role of tie-breaking rules, LiCalzi and Pavan [2005] and Burkett and Woodward [2020b] at elastic supply, McAdams [2007] at commitment, and Woodward [2019a] and Burkett and Woodward [2020a] at the role of price selection.

Our bid representation theorem is closest to that of Swinkels [2001], who studies pay-as-bid and uniform-price auctions in large markets, and is also similar to Holmberg [2009].[20] In the limit, as the number of bidders goes to infinity our representation converges to that obtained in Swinkels [2001]; our contribution lies in establishing the representation of bids as averages of marginal values in all finite markets and not only in the limit. That such a bid representation remains valid in finite markets is surprising in the context of prior literature, which can be naturally read as suggesting that pay-as-bid equilibria are complex in environments for which we establish our bid reprentation: Prior constructions of finite-market equilibria focused on the setting in which bidders' marginal values are linear in quantity and the distribution of supply is (a special case of) the generalized Pareto distribution; see Wang and Zender [2002], Federico and Rahman [2003], Hästö and Holmberg [2006], Holmberg [2009], Ewerhart et al. [2010], and Ausubel et al. [2014]. This literature expressed equilibrium bids in terms of the intercept and slope of the linear demand and the parameters of the generalized Pareto distribution. In addition to analyzing the linear-Pareto setting, Holmberg [2009] derives a closed-form representation for symmetric and smooth equilibria subject to constraints on supply. We make no such assumptions, and instead prove that equilibria are symmetric and smooth; our results therefore provide support for their analysis. Also, as mentioned in our comparison to Swinkels [2001], our finite-market representation of bids as weighted averages of marginal values is new.

Our transparency result—that deterministic selling strategies are optimal—may appear familiar from the no-haggling theorem of Riley and Zeckhauser [1983]. However, in multi-object settings the reverse has been shown by Pycia [2006] and Manelli and Vincent [2006]; and, as mentioned above, nondeterministic supply may have a role in uniform-price auctions. Furthermore, there is a subtlety specific to pay-as-bid that might suggest a role for randomization: by randomizing supply below the monopoly quantity, the seller forces bidders to compete and bid more for these quantities, and in pay-as-bid the seller collects the raised

[20] We focus our discussion on settings with decreasing marginal utilities; for constant marginal utilities see Back and Zender [1993], and Ausubel et al. [2014] among others.

bids even when the realized supply is near the monopoly quantity. We show that, despite these considerations, committing to deterministic supply is indeed optimal.[21]

We also establish a revelation result: independent of the parameterization of the auction, the seller prefers to commit to announce the realization of supply prior to bid submission. The reason for this revelation result (as well as the preceding transparency result) is a novel bound on revenues in random pay-as-bid auctions rather than Milgrom and Weber's (1982) celebrated linkage principle; the linkage principle is known to fail in the multi-unit auction context, cf. Perry and Reny [1999]. Furthermore, while our setting is one of Bayesian persuasion and information design, the full revelation we establish stands in stark contrast to Kamenica and Gentzkow's (2011) paradigmatic insight that in information design and Bayesian persuasion, the sender wants to withhold—or obfuscate—information. Related to information design, Bergemann et al. [2017] and Bergemann et al. [2019] also find the optimality of withholding information in single-unit auctions. The reason why there is no contradiction between us establishing the optimality of full revelation while these papers find a rationale for withholding is that their sender posseses information about bidders' valuations while in our analysis the bidders (receivers) are fully informed of their own value functions and the seller (sender) has and can release information about the quantity supplied, which is a key element of the bidders' strategic interaction.[22]

While we are not aware of prior literature on optimal design of supply and reserve prices in pay-as-bid, the general mechanism design question was addressed by Maskin and Riley [1989]: what is the revenue-maximizing mechanism to sell divisible goods? The optimal mechanism they described is complex and in practice the choice seems to be between the

[21]We further establish that the seller sets deterministic supply in all Perfect Bayesian Equilibria of the game in which the seller first designs a pay-as-bid auction and then the bidders bid; this is in contrast to uniform-price acution design games for which we construct equilibria in which the seller sets random supply (even though we show that the revenue-maximal equilibrium has deterministic supply also in the uniform-price auction). Note that Chen et al. [2019] show that individual outcomes of a given random mechanism can be replicated by a deterministic mechanism when there are multiple privately informed participants, while we show that not only can the maximal revenue generated by any random pay-as-bid auction be obtained by some deterministic mechanism, but also that this is possible without fundamentally changing the auction mechanism.

[22]In single-unit auctions bidders necessarily have full information regarding the quantity supplied, and the auctioneer's role in information design is inherently limited. Fang and Parreiras [2003] and Board [2009] study the limits of the linkage principle and the resulting benefits of information withdrawal or obfuscation. Obfuscation is also established in other settings in which—like in our auction setting—the sender's interest (more revenue) is fundamentally misaligned with the bidders' interests (reducing payment); in a global games context see, e.g., Li et al. [2020]. For analysis of bidders' investement in information acquisition in auctions see e.g. Persico [2000] who finds that bidders in first-price auctions acquire more value-relevant information than bidders in second-price auctions. Finally, while we study a seller/sender who is able to commit to a revelation strategy, our revelation result immediately implies that a sender unable to commit would also fully reveal supply information. For information revelation under no commitment see e.g. Grossman and Hart [1980] and Milgrom [1981].

much simpler auction mechanisms: pay-as-bid and uniform-price. On the other hand, design issues have been addressed in the context of uniform-price auction; see our discussion of collusion and collusive-seeming equilibria above.[23]

Our revenue and welfare comparisons between pay-as-bid and uniform-price auctions contribute to the rich discussion of the pros and cons of these two formats. Swinkels [2001] showed that pay-as-bid and uniform-price are revenue- and welfare-equivalent in large markets; our equivalence result does not rely on the size of the market. Wang and Zender [2002] find pay-as-bid revenue superior in the equilibria of the complete-information linear-Pareto model their consider, and Woodward [2019b] extends this dominance to mixed-price combinations of pay-as-bid and uniform-price auctions. Ausubel et al. [2014] show that—with ex-ante asymmetric bidders with flat demands—either format can be revenue superior.[24] Our results on approximate revenue equivalence with small informational asymmetries complement this ambiguity: for uniform-price to raise significantly more revenue than pay-as-bid, bidders must be significantly asymmetric. In aggregate, prior theoretical work on the pay-as-bid versus uniform-price question has focused on revenue comparisons for fixed supply distributions and has allowed for neither reserve price nor supply optimization; indeed, the previous studies of pay-as-bid auctions with decreasing marginal values employed parametric specifications that did not support the analysis of design questions. Thus these results cannot address whether a well-designed pay-as-bid auction is preferable to a well-designed uniform-price auction. We go beyond these earlier papers both by allowing seller's optimization and by imposing no assumptions on seller's information about the bidders.

Our divisible-good optimal revenue equivalence result provides a benchmark for the long-standing empirical debate whether pay-as-bid or uniform-price auctions raise higher expected revenues. This debate has attracted substantial empirical attention, with Hortaçsu and McAdams [2010] finding no statistically significant differences in revenues, Février et al. [2002], Kang and Puller [2008], and Marszalec [2017] finding slightly higher revenues in pay-as-bid, and Castellanos and Oviedo [2004], Armantier and Sbaï [2006], and Armantier and Sbaï [2009] finding slightly higher revenues in uniform-price. Hortaçsu et al. [2018] argue that the revenues are similar.[25] As noted above, many of these papers conduct a counterfac-

[23] Furthermore, in the environments we focus on, the bidders' private information is correlated and hence the seller can nearly extract their full surplus using Cremer-McLean-type mechanisms [Crémer and McLean, 1988]; cf. footnote 61.

[24] With symmetric or non-flat demands bidders pay-as-bid is revenue superior in all examples they consider. The special supply distributions these papers consider are not revenue-maximizing, hence there is no conflict between their strict rankings and our revenue equivalence. For analysis of revenue comparisons with flat demands see also, e.g., A.W.Anwar [1999] and Engelbrecht-Wiggans and Kahn [2002].

[25] They note that bids in U.S. Treasury auctions are typically "flat" and infer that not much surplus is retained by bidders; an alternative explanation highlighted by our analysis is that the bidders are close to being completely informed. Note also that while flatness implies that there is not much difference between

tual estimation of uniform-price revenues assuming truthful bidding, which is precisely the equilibrium selection under which our theoretical revenue and welfare equivalence obtains.[26]

Our results regarding the selection of auction format have other empirical implications. We show in our analysis of the auction design game that the auctioneer either strictly prefers a pay-as-bid auction or is indifferent between the pay-as-bid and uniform-price formats. All else equal, our model suggests that pay-as-bid auctions should be more prevalent than uniform-price auctions. This claim is supported by the multi-country analysis of security auction implementation in Brenner et al. [2009], which finds that pay-as-bid auctions are implemented by more than twice as many nations as implement uniform-price auctions, as well as the analysis of electricity markets in Del Río [2017], which finds that pay-as-bid auctions represent nearly 90% of electricity auctions. Additionally, counterfactual analysis of uniform-price auctions assumes truthful reporting of values to obtain an upper bound on unobserved revenue. Our results can be taken to show that this bound is tight only if bidders are playing a seller-optimal equilibrium; otherwise there may be a significant divergence between observed revenue and counterfactual predictions. Additionally, since revenue-dominance of the pay-as-bid auction implies that a seller should implement the uniform-price format only if she expects this favorable equilibrium to be played, we should expect counterfactual analysis from witnessed uniform-price auctions to find approximate revenue equivalence.[27]

Finally, let us note that our analysis of pay-as-bid auctions can be reinterpreted as a model of dynamic oligopolistic competition among sellers who at each moment of time compete a la Bertrand for sales and who are uncertain how many more buyers are yet to arrive. Prior sales determine the production costs for subsequent sales, thus the sellers need to balance current profits with the change in production costs in the future. This methodological link between pay-as-bid auctions and dynamic oligopolistic competition is new, and we develop it in follow up work.[28]

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the revenues generated by the pay-as-bid and uniform-price auctions, the uniform-price auction brings large downside potential in the form of collusive-seeming behavior. Our uniqueness theorem shows that the pay-as-bid format mitigates this risk.

[26] We show that revenue is maximized in the uniform price auction when the seller offers a deterministic quantity for sale, and bidders bid their true marginal values for the quantity they obtain. Whether they bid their true marginal values for quantities they do not obtain is irrelevant in equilibrium, thus we refer to truthful bidding at the received quantity as truthful bidding. This assumption is satisfied in the counterfactual approaches, where a bid curve is truthful if it matches the bidder's marginal value curve.

[27] As discussed earlier, empirical analysis of uniform-price auctions is hampered by the equilibrium selection problem. We know of no existing analyses comparing witnessed uniform-price auctions to counterfactual pay-as-bid auctions.

[28] The oligopolistic sellers uncertain of future demands correspond to bidders in the pay-as-bid auction, and sellers' costs correspond to bidders' values. For prior studies of dynamic competition see e.g. Deneckere and Peck [2012]; while they study competition among a continuum of sellers, the pay-as-bid-based approach

2 Model

There are $n \geq 2$ bidders, $i \in \{1, ..., n\}$. Each bidder's marginal valuation for quantity q is denoted $v^i(q; s_i) = v(q; s_i)$, where s_i is a signal privately known to bidder i. We assume that $v(\cdot; s)$ is strictly decreasing, Lipschitz continuous, and almost-everywhere differentiable in q. We allow arbitrary dimensionality of s, and an arbitrary integrable $v(q; \cdot)$. The seller is uninformed and we study environments in which s_i are highly correlated across bidders: in Sections 3-5 we analyse the case when the correlation is perfect that is $s_1 = ... = s_n = s$, without imposing any further assumptions on the distribution of s; in Section 6 we relax the perfect correlation assumption and show that our insights are robust to this relaxation. Under perfect correlation, signal s has no strategic importance for bidders participating an auction, and thus when studying the equilibrium among such bidders in Section 3, we fix s and denote the bidders' marginal valuation by $v^i(q) = v(q)$. Bidders' information plays an important role in the analysis of the seller's problem in Sections 4, 5, and 6.[29]

To simplify the exposition of the design problem, we normalize the seller's cost to 0. Our insights do not hinge on this normalization, and remain valid for any convex cost function.[30] Our design analysis builds on the existence, uniqueness, and bid representation results in equilibrium of the pay-as-bid auction. We thus start by analysing such equilibria. In the equilibrium analysis we study supply Q drawn from distribution F with density $f > 0$ and support $[0, \overline{Q}]$; we also allow F with full mass concentrated at one point. Q is independent of the bidders' signal s.[31] Otherwise we impose no global assumptions on F. We denote the inverse hazard rate by $H = \frac{1-F}{f}$.

In the pay-as-bid auction, each bidder submits a weakly decreasing bid function $b^i(q) : [0, \overline{Q}] \rightarrow \mathbb{R}_+$. Without loss of generality we may assume that the bid functions are right-continuous.[32] The auctioneer then sets the market price p (also known as the stop-out

allows for the strategic interaction between a finite number of sellers. The other canonical multi-unit auction format, the uniform-price auction, was earlier interpreted in terms of static oligopolistic competition by Klemperer and Meyer [1989].

[29]The seller may not know the bidders' information if, for example, the seller needs to commit to the auction mechanism before this information is revealed. Alternatively, the seller may want to fix a single design for multiple auctions.

[30]The reason why more general cost functions do not substantively change the analysis is that the Transparency Theorem, on which the analysis of design builds, is valid irrespective of seller's cost function. We provide more detailed discussion in Section 4.

[31]This last assumption is not needed in our analysis of elastic supply (see Appendix A). Q might be an on-path or off-path supply in seller's design problem or it might represent e.g. supply net of non-competitive bids as discussed in Back and Zender [1993], Wang and Zender [2002], and subsequent literature.

[32]This assumption is without loss because we study a perfectly-divisible good and we ration quantities pro-rata on the margin. Indeed, we could alternatively consider an equilibrium in strategies that are not necessarily right-continuous. By assumption, the equilibrium bid function of a bidder is weakly decreasing, hence by changing it on measure zero of quantities we can assure the bid function is right continuous. Such a

price),

$$p = \sup \left\{ p' : \; q_1 + ... + q_n \geq Q \text{ for all } q_1, \ldots, q_n \text{ such that } \; b^1(q_1), ..., b^n(q_n) \leq p' \right\}.$$

If the set over which the supremum is taken is empty, then the stop-out price is set to the reserve price R; we start in Section 3 with $R = 0$ and then extend the analysis to any positive reserve price. Agents are awarded a quantity associated with their demand at the stop-out price,

$$q_i = \max \left\{ q' : b^i(q') \geq p \right\},$$

as long as there is no need to ration them. When necessary, we ration pro-rata on the margin, the standard tie-breaking in divisible-good auctions. The details of the rationing rule have no impact on the analysis of equilibrium bidding we pursue in Section 3.[33] The demand function (the mapping from p to q^i) is denoted by $\varphi^i(\cdot)$.[34] Agents pay their bid for each unit received, and utility is quasilinear in monetary transfers; hence,

$$u^i\left(b^i\right) = \int_0^{q^i(p)} v(x) - b^i(x)\, dx.$$

We study Bayesian-Nash equilibria in pure strategies.

3 Pay-as-Bid Equilibrium

We start our analysis by establishing novel and general results for the benchmark case when bidders are symmetrically informed that is $s_1 = ... = s_n = s$. We relax this assumption in Section 6. When analyzing an equilibrium of the Pay-as-Bid auction, signal s has no strategic importance for bidders and thus we fix s and denote the bidders' marginal valuation by $v^i(q) = v(q)$.

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change has no impact on this bidder's profit, or on the profits of any of the other bidders, because rationing pro-rata on the margin is monotonic in the sense of footnote 33. In fact, there is no impact on bidders' profits even conditional on any realization of Q.

[33] The only place when we rely on rationing rule is the analysis of reserve prices but even in this part of the analysis all we need is that rationing rule is monotonic: that is, the quantity assigned to each bidder increases when the stop-out price decreases; rationing pro-rata on the margin satisfies this property.

[34] Where $b^i(\cdot)$ is constant, φ^i is not well-defined. Where important, we will use $\underline{\varphi}^i$ and $\overline{\varphi}^i$ to denote the right- and left-continuous (respectively) inverses of b, $\underline{\varphi}^i(p) = \sup\left\{q : b^i(q) > p\right\}$ and $\overline{\varphi}^i(p) = \sup\left\{q : b^i(q) \geq p\right\}$.

3.1 Existence, Uniqueness, and The Bid Representation Theorem

We show that equilibrium is unique and tractable; then the existence of equilibrium can be analyzed in terms of what equilibrium strategies must be, if an equilibrium exists. The condition for equilibrium existence is stated in terms of these strategies, therefore we defer discussion of existence until after our uniqueness and representation results. For expositional simplicity all results (except for Theorem 3 on equilibrium existence) are given conditional on the existence of Bayesian-Nash equilibrium. Furthermore, our uniqueness and representation results constrain attention to relevant quantities at which bids can possibly affect utility; bids for quantities which the bidder never receives must be weakly decreasing and sufficiently competitive, but are not typically uniquely determined.[35] Proofs of all results may be found in the Appendix.

Theorem 1. [Uniqueness] *The Bayesian-Nash equilibrium is unique.*

Equilibrium uniqueness leads us to our main insight, the bid representation theorem:

Let us now introduce the notion of weighting distributions. For any quantity $Q \in [0, \overline{Q})$, the n-bidder weighting distribution of F has c.d.f. $F^{Q,n}$ that increases from 0 when $x = Q$ to 1 when $x = \overline{Q}$. This c.d.f. is given by

$$F^{Q,n}(x) = 1 - \left(\frac{1 - F(x)}{1 - F(Q)} \right)^{\frac{n-1}{n}}.$$

The auxiliary c.d.f.s $F^{Q,n}$ play a central role in our bid representation theorem below. These distributions depend only the number of bidders and the distribution of supply, and not on any bidder's true demand. As the number of bidders increases the weighting distributions put more weight on lower quantities.

Theorem 2. [Bid Representation Theorem] *The unique equilibrium is symmetric and the bid b^i of each bidder i is given by*

$$b^i(q) = \int_{nq}^{\overline{Q}} v\left(\frac{x}{n}\right) dF^{nq,n}(x). \tag{1}$$

[35] The reason a bidder's bids on never-won quantities need to be sufficiently competitive is to ensure that other bidders do not want decrease their bids on relevant quantities. In the setting with reserve prices, which we analyze in Section 5, the bids on never-won quantities may not need to be competitive and hence these bids are even less determined, but the equilibrium bids on the relevant quantities, that is, those which are sometimes marginal, remain uniquely determined. Importantly, these bids being insufficiently competitive does not induce alternate equilibria: there are no equilibria in which these bids are lower than required to support the unique equilibrium we find.

The resulting market price function is given by

$$p(Q) = \int_{Q}^{\overline{Q}} v\left(\frac{x}{n}\right) dF^{Q,n}(x). \tag{2}$$

Recall that we impose no assumptions on symmetry of equilibrium bids, their strict monotonicity, nor continuity or differentiability; we derive all these properties. Furthermore, the equilibrium bids b^i are appropriately-weighted averages of bidders' marginal values v, and in this they resemble the bids in first price auctions with privately informed bidders. Because the unique equilibrium is symmetric, the market price $p(Q)$ given supply Q and the bid functions b^i are related in a natural way, $b^i(q) = p(nq)$.

Consider three examples. Substitution into our bid representation shows that when marginal values v are linear and the supply distribution F is generalized Pareto, $F(x) = 1 - \left(1 - \frac{x}{\overline{Q}}\right)^{\alpha}$ for some $\alpha > 0$, the equilibrium bids are linear in quantity. This case of our general setting has been analyzed by Ewerhart et al. [2010], and Ausubel et al. [2014].[36] Our bid representation remains valid when F puts all its mass on $\overline{Q}$: taking the limit of continuous probability distributions which place increasingly more probability near $\overline{Q}$, the representation implies that equilibrium bids are flat, as they should be. Finally, Figure 1 illustrates the equilibrium bids for ten bidders with linear marginal values who face a distribution of supply that is truncated normal. This and the subsequent figures represent bids, marginal values, and the c.d.f. of supply; it is easy to distinguish between the three curves since bids and the marginal values are decreasing (and bids are below marginal values) while the c.d.f. is increasing.[37]

Theorem 3. [Existence] *There exists a pure-strategy Bayesian-Nash equilibrium whenever for any* $q \in [0, \overline{Q}]$, $(v(q) - b)(1 - F(q + (n-1)\varphi(b))$ *is monotonic or single-peaked on* $b \in [p(\overline{Q}), p(\max\{\overline{Q}, nq/(n-1)\})]$.

In other words, the equilibrium exists if at each relevant quantity q the bid $b_i(q)$ satisfies the standard second order condition of bidder i's qunatity-by-quantity (pointwise) utility optimization. The result follows from familiar arguments in single-dimensional contexts. Ig-

[36]They calculated bid functions in terms of the parameters of their model (linear marginal values and Pareto distribution of supply) and do not rely on or recognize the representation of bids as weighted averages that is crucial to our subsequent analysis. While we focus on bounded distributions (as did Ewerhart et al., 2010), Ausubel et al. [2014] look at both bounded and unbounded Pareto distributions, and Wang and Zender [2002] and Holmberg [2009] look at unbounded Pareto distributions. Our general approach remains valid for unbounded distributions, including Pareto, except that uniqueness requires a lower bound on admissible bids, e.g. an assumption that bids are nonnegative. We provide more details on this extension of our results in our discussion of reserve prices.

[37]In all figures, we check our equilibrium existence condition and calculate bids numerically using R. In Figure 1 we use a normal distribution with mean 3 and standard deviation 1, truncated to the interval $[0, 6]$.

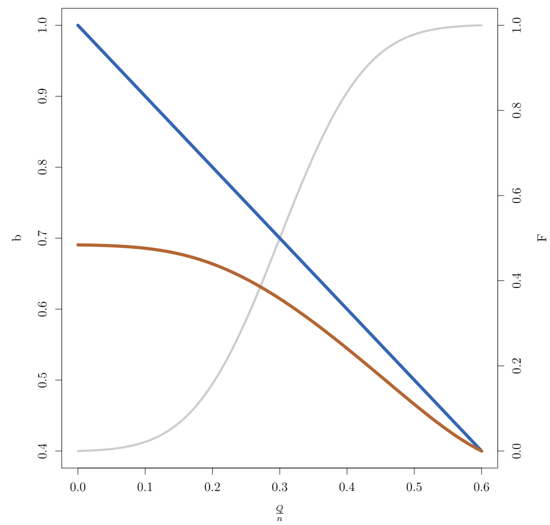

Figure 1: Equilibrium bids when the distribution of supply Q is truncated normal.

noring the monotonicity constraint, if a bid function solves the bidder's optimization pointwise then it is a global maximum and a best response. Given a quantity q, if the pointwise objective is single-peaked on the range of feasible prices there is at most one bid at which the best response first-order condition is satisfied. Since bids in the symmetric equilibrium given in Theorem 2 solve these first-order conditions, they are best responses. Note that Theorem 2 implies that φ is expressible in closed form in terms of model fundamentals, so Theorem 3 is not conditioning equilibrium existence on an equilibrium object. We provide a proof and additional comments on our existence condition in Appendix F.

Consider some examples. Our sufficient condition is satisfied when marginal values v are linear and F is a uniform distribution or a generalized Pareto distribution, $F(x) = 1-\left(1-\frac{x}{\overline{Q}}\right)^{\alpha}$ where $\alpha > 0$.[38] When marginal values have slope bounded away from zero, this condition is also satisfied for any twice-differentiable c.d.f. F provided there are sufficiently many bidders.[39] And, the sufficient condition is satisfied whenever the inverse hazard rate H is increasing—hence when the hazard rate is decreasing—irrespective of the marginal value

[38]The existence of equilibrium in the linear/generalized Pareto example was independently established by Ewerhart et al. [2010] and Ausubel et al. [2014]. In Section 3.3, we extend our results to unbounded distributions, including the unbounded Pareto distributions studied by Wang and Zender [2002], Federico and Rahman [2003], and Holmberg [2009]; our sufficient condition remains satisfied for unbounded Pareto distributions.

[39]This is easiest to see in the weaker but more technical condition preceding the proof of Theorem 3 in Appendix F. Holding F constant, if there is a (Q, p) at which the implication is violated then if there are $n' < n$ bidders there is a (Q', p') such that the condition is also violated, since $p(\cdot)$ is increasing in n. Holding per-capita supply constant, $Y_q \to 0$ for almost every Q, so when $v_q > \varepsilon > 0$ the implication will be satisfied.

function v.[40] This follows since the left- and right-hand terms of the objective are decreasing in b (φ is decreasing in b). In the sequel we illustrate our other results with additional examples in which a pure-strategy equilibrium exists.[41]

While our sufficient condition shows that the equilibrium exists in many cases of interest, there are situations in which the equilibrium does not exist; see the discussion in our introduction.[42]

The rest of our paper builds upon the above results to establish qualitative properties of the unique equilibrium, and to provide guidance as to how to design divisible good auctions including for environments in which bidders' information is not perfectly correlated.

3.2 Equilibrium Properties and Comparative Statics

The bid representation of Theorem 2 has many natural implications. We discuss them in this subsection as well as provide the resulting link between revenue and the distribution of private information in Section 4.

3.2.1 Flat Bids, Low Margins, and Concentrated Distributions

A case of particular interest arises when the distribution of supply is concentrated near some target quantity. We say that a distribution is δ-concentrated near quantity Q^* if $1-\delta$ of the mass of supply is within δ of quantity Q^*.

Our bid representation theorem implies that the bids on initial quantities are nearly flat for concentrated distributions.

Corollary 1. [Flat Bids] *For any $\varepsilon > 0$ and quantity Q^* there exists $\delta > 0$ such that, if the supply is δ-concentrated near Q^*, then the equilibrium bids for all quantities lower than $\frac{Q^*}{n} - \varepsilon$ are within ε of $v\left(\frac{Q^*}{n}\right)$.*

Figure 2 illustrates the flattening of equilibrium bids; in the three sub-figures ten bidders face supply distributions that are increasingly concentrated around the total supply of 6 (per capita supply of 0.6).

[40] The sufficiency of decreasing hazard rate for equilibrium existence was established by Holmberg [2009].

[41] The first of these examples features a normal distribution, the second one features strictly concave marginal values, and the last one features reserve prices. In general, our existence condition is closed with respect to several changes of the environment: adding a bidder preserves existence, making the marginal values less concave (or more convex) preserves existence, and imposing a reserve price preserves existence.

[42] The construction of a tighter existence condition is complicated by the possibility of monotonicity-constrained deviations from the symmetric solution to the market clearing equation provided in Theorem 2. A global best response might exist which is the aggregation of nonoptimal local behavior. Our sufficient condition implies that the optimization problem is single-peaked in bid, and there is a unique global optimum.

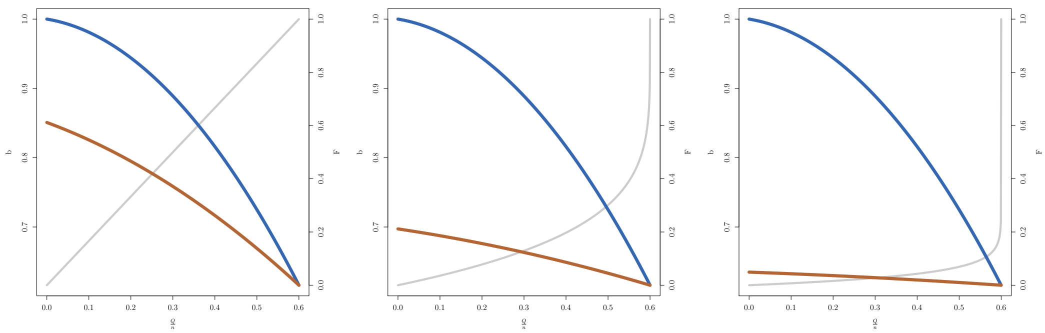

Figure 2: Bids are flatter for more concentrated distributions of supply.

Our representation theorem has also implications for bidders' margins. In the corollary below we refer to the supremum of quantities the bidder wins with positive probability as the highest quantity a bidder can win in equilibrium.

Corollary 2. [**Low Margins**] *The highest quantity a bidder can win in equilibrium is* $\frac{1}{n}\overline{Q}$*, and the bid at this quantity equals the marginal value,* $b\left(\frac{1}{n}\overline{Q}\right) = v\left(\frac{1}{n}\overline{Q}\right)$*. Furthermore, for any* $\varepsilon > 0$ *and quantity* Q^* *there exists* $\delta > 0$ *such that, if supply is* δ*-concentrated near* Q^**, then each bidder's equilibrium margin* $v\left(\frac{1}{n}Q^* - \delta\right) - b\left(\frac{1}{n}Q^* - \delta\right)$ *on the* $\frac{1}{n}Q^* - \delta$ *unit is lower than* ε*.*

Thus, each bidder's margin on the last unit they could win is zero; and, if the supply is concentrated around some quantity Q^*, then the margin on units just below $\frac{1}{n}Q^*$ is close to zero.

3.2.2 Comparative Statics

Our bid representation theorem allows us to easily deduce how bidding behavior changes when the environment changes. First, as one could expect, an increase in marginal values always benefits the seller: higher values imply higher revenue.

Corollary 3. [**Higher Values**] *If bidders' marginal values increase, the seller's revenue goes up.*

The bid representation theorem further implies that if there is an affine transformation of bidders' marginal values from v to $\alpha v + \beta$, then the seller's revenue changes from π to $\alpha\pi + \beta$. In particular, all the additional surplus goes to the seller when the value of all bidders is raised by a constant.

Also as one would expect, the bidders' equilibrium margins are lower and the seller's revenue is larger when there are more bidders and the distribution of supply is held constant:

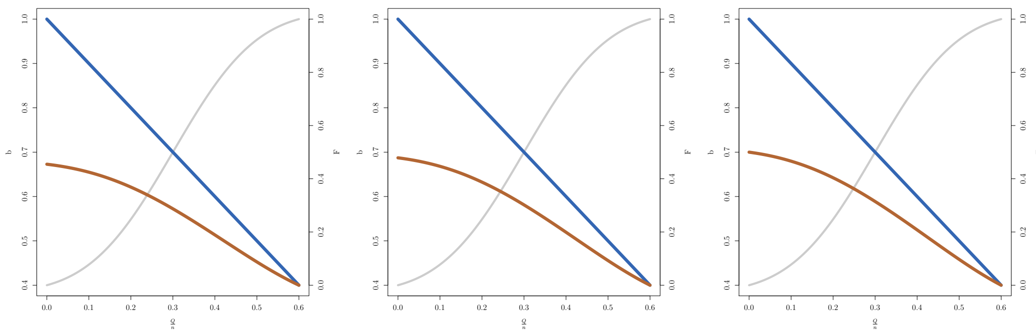

Figure 3: Bids go up when more bidders arrive (and per capita quantity is kept constant) but not by much: 5 bidders on the left, 10 bidders in the middle, and 5 million bidders on the right. Note that all axis scales are identical.

Corollary 4. [More Bidders] *The bidders submit higher bids and the seller's revenue is larger when there are more bidders (both when the supply distribution is held constant, and when the per-capita supply distribution is held constant).*

Indeed, as the number of bidders increases, $1 - F^{Q,n}(x) = \left(\frac{1-F(x)}{1-F(Q)}\right)^{\frac{n-1}{n}}$ decreases, and hence $F^{Q,n}(x)$ increases, thus the mass is shifted towards lower x, where marginal values are higher. At the same time, the marginal value at x either increases in n (if we keep the distribution of supply constant) or stays constant (if we keep the distribution of per-capita supply constant). Both these effects point in the same direction, implying that the bids and expected revenue increase in the number of bidders. This argument also shows that the seller's revenue is increasing if we add bidders while proportionately raising supply, and that a bidder's profits are decreasing in the number of bidders even if we keep per-capita supply constant.

While bidders raise their bids when facing more bidders even if the per-capita distribution stays constant, our bid representation theorem implies that the changes are small.[43] This is illustrated in Figure 3 in which increasing the number of bidders from 5 bidders to 10 bidders has only a small impact on the bids, as does the further increase from 10 bidders to 5 *million* bidders. To see analytically what happens for large numbers of bidders, let us denote the distribution of per-capita supply by $\bar{\bar{F}}$,

$$\overline{F}(x) = F(nx),$$
$$\overline{F}^{q,n}(x) = F^{nq,n}(nx).$$

[43]Notice that if we keep the supply distribution fixed while more and more bidders participate in the auction, then in the large market limit the revenue converges to average supply times the value on the initial unit. See Swinkels [2001].

and note that $b(q) = \frac{1}{n}\int_q^{\frac{1}{n}\overline{Q}} v(x)\, d\overline{F}^{q,n}(x)$. As $n \to \infty$ we get $\overline{F}^{q,n}(x) \to \frac{\overline{F}(x)-\overline{F}(q)}{1-\overline{F}(q)}$, and the limit bids take the form

$$b(q) = \frac{1}{n}\int_q^{\frac{1}{n}\overline{Q}} v(x)\, d\frac{\overline{F}(x)-\overline{F}(q)}{1-\overline{F}(q)} = \frac{1}{n}\left(\frac{1}{1-\overline{F}(q)}\right)\int_q^{\frac{1}{n}\overline{Q}} v(x)\, d\overline{F}(x).$$

In particular, in large markets, the bid for any unit is given by the average marginal value of higher units, where the average is taken with respect to (scaled) per-capita supply distribution.

3.3 Equilibrium with Reserve Prices

As a preparation for the design analysis we now show that our characterization of the equilibrium of the bidders' game readily extends to pay-as-bid with a reserve price.

Theorem 4. [Equilibrium with Reserve Prices] *In the pay-as-bid auction with reserve price R, the equilibrium is unique and is identical to the unique equilibrium in the pay-as-bid auction with type-dependent supply distribution $F^R(Q;s) = F(Q)$ for $Q < \hat{Q}(s)$ and $F^R\left(\hat{Q};s\right) = 1$, where $\hat{Q}(s) = nv^{-1}(R;s)$.*

Notice that distribution F^R has a probability mass at supply $\hat{Q}$, which is the largest supply under this distribution. While we derived our results for atomless distributions, our arguments would not change if we allowed an atom at the highest supply. Thus, all our equilibrium results remain applicable.[44] Furthermore, notice that we developed the theory of equilibrium bidding assuming that the distribution of supply is bounded. However, in the presence of a reserve price, any unbounded distribution is effectively bounded, hence the boundedness assumption may be relaxed.

The rest of the proof of Theorem 4 is then simple. When the distribution of supply is F^R then the last relevant bid is exactly R by Corollary 2, and hence imposing the reserve price of R does not change bidders' behavior. Furthermore the equilibrium bids against F^R remain equilibrium bids against F with reserve price R, and one direction of the Theorem is proven. Consider now equilibrium bids in an auction with reserve price R.[45] The sum of bidders' demands is then always weakly lower than $nv^{-1}(R)$ and hence their bids constitute

[44] We provide more details in the Appendix. Importantly, if marginal values v and a distribution of supply F satisfy our sufficient condition for equilibrium existence, then F^R satisfies this condition as well. Indeed, for $Q < \hat{Q}$, folding the tail of the distribution F into an atom in F^R leaves the left-hand side of this condition unchanged while making the right-hand side more negative (since its numerator is negative and the mass shift makes the positive denominator smaller).

[45] Instead of this step of the argument, we could check directly that our uniqueness result, Theorem 4, remains true in the setting with reserve prices.

an equilibrium when the supply is distributed according to F^R. This establishes the other direction of Theorem 4.

4 Designing Pay-as-Bid Auctions

In this section we maintain the assumption that the pay-as-bid format is run and we analyze the design of such auctions focusing on the reserve price and the distribution of supply, the two natural elements of pay-as-bid auction that the seller can select.[46] In Section 5, we leverage our analysis of design to study the choice between two main multi-unit auction mechanisms: pay-as-bid and uniform-price.

In this section, we consider the case in which the bidders still all observe the same signal s but we allow the seller not knowing the bidders' signal s and having a belief about it, $s \sim \sigma$. In Section 6 we show that our insights are robust to the introduction of small informational asymmetry among the bidders. As design decisions are taken from the seller's perspective, our terminology in this and the next sections explicitly keeps track of the bidders' information. We impose no assumptions on the distribution σ other than $v(q; \cdot)$ being integrable.

4.1 Transparency and Revelation

The key insight that underlies our design analysis is that—in contrast to typical multidimensional mechanism design problems discussed in the Introduction—in an optimized pay-as-bid auction deterministic—and, hence, transparent—supply is optimal. Furthermore, if supply is exogenously random, then it is optimal for the seller to both set a determistic supply cap and to announce the realized supply to the bidders prior to the auction.

First, suppose that the seller has some determistic quantity $\overline{Q}$ of the good; we relax this assumption below. For any fixed reserve price, we consider the problem of designing a supply distribution F that maximizes the seller's revenue. The seller has the option to offer a stochastic distribution over multiple quantities, and it is plausible that such randomization could increase his expected revenue. Offering randomization over quantities larger than the optimal deterministic supply $Q^{\star}(R)$ may be relatively easily shown to be suboptimal: the seller's profit on the units above $Q^{\star}(R)$ is lower than his profit on deterministically selling $Q^{\star}(R)$, and moreover offering quantities above $Q^{\star}(R)$ suppresses the bids submitted for

[46] In Appendix A, we show that our design insights extend to the environment in which the seller is not restricted to setting a reserve price and supply distribution and instead the seller cand set a joint distribution for both.

$Q^\star(R)/n$. On the other hand, offering quantities lower than $Q^\star(R)$ offers the seller a trade-off: he sometimes sells less than $Q^\star(R)$, with a direct and negative revenue impact, but when he sells quantity $Q^\star(R)$ he will receive higher payments due to the pay-as-bid nature of the auction. The issue is well illustrated in Figure 2, in which concentrating supply lowers the bids.[47]

We show that selling the deterministic supply $Q^\star$ is in fact revenue-maximizing for sellers across all pure-strategy equilibria; for this reason in the sequel we refer to $Q^\star$ as optimal supply. In this section, we restrict attention to pure strategy equilibria and, relatedly, maintain our global restrictions on the support of supply. In Appendix A, we relax these and other restrictions—e.g., allowing elastic supply—and prove that the pure-strategy equilibrium under transparent and deterministic supply revenue dominates any mixed-strategy equilibrium at any random supply.[48]

Theorem 5. [Transparency of Optimal Supply] *In pure-strategy equilibria, the seller's revenue under non-deterministic supply is strictly lower than her revenue under optimal deterministic supply. Optimal deterministic supply is given by the solution to the monopolist's problem when facing uncertain demand.*

As the following proof sketch indicates, Theorem 5 remains valid if the reserve price is arbitrary rather than optimized. The transparency result also remains valid for sellers who maximize profits equal to revenue net of costs, provided the marginal cost is weakly increasing.[49] Such sellers optimally choose the deterministic quantity (or quantity cap) that maximizes the expected revenue minus cost rather than the quantity that maximizes the expected revenue. Taking the cost into account affects what quantity is optimal, but it does not change the point that optimal supply is deterministic.

To prove this theorem, we start with an arbitrary reserve price and supply distribution and the induced pure-strategy equilibrium bids. Holding equilibrium bids fixed, we use our equilibrium analysis (Theorems 2 and 4) to bound expected revenue by the standard

[47] A priori such trade-offs can go either way; see Pycia [2006] and the Introduction.

[48] The restriction to pure-strategy equilibria can be also straightforwardly relaxed in the special case in which bidders' have no private information. Furthermore, as pay-as-bid is largely employed by central banks and governments, the efficiency of allocations may be an important concern and a reason a seller may want to ensure that pure-strategy equilibrium is being played. The symmetry of equilibrium strategies we prove in Theorem 2 implies that in pure-strategy equilibrium the marginal value for any unit received is higher than the marginal value for any unit not received. In a pure-strategy equilibrium, there are thus no efficiency improving re-allocations of units among bidders; this property trivially fails in any mixed-strategy equilibrium that is not essentially in pure strategies.

[49] In the absence of the increasing marginal cost assumption, an analogue of Theorem 5 would need to be modified to take account of resulting ironing. See also our remark at the end of the proof of the theorem in Appendix I.

monopoly revenue given the supply distribution.[50] In effect we obtain the following bound on the expected revenue,

$$\mathbb{E}_{s,Q}\left[\pi\left(s,Q\right)\right] \leq \int_{0}^{\overline{Q}} \mathbb{E}_{s}\left[\pi^{m}\left(s,Q\right)\right] dF\left(Q\right), \tag{3}$$

where $\pi\left(s,Q\right)$ is the seller's revenue when bidders' signal is s, the realization of supply is Q, and bidders bid againts the distribution of supply F while $\pi^{m}\left(s,Q\right)$ is the seller's revenue when bidders' signal is s, the realization of supply is Q, and bidders bid againts the distribution of supply that puts probability 1 on supply quantity Q.[51] This upper bound implies that the seller's revenue is maximized when the seller sets the supply to be always equal to the revenue-maximizing deterministic supply. We provide the details of the proof in Appendix I (bound (3) above restates inequality (15) in the proof).

The above structure of the proof has two important implications. First, under the additional restriction that $Q\mathbb{E}_{s}[v^{-1}(Q;s)]$ is single-peaked in Q, the proof of Theorem 5 is applicable to environments in which the seller's underlying supply is random and the seller can lower the supply but cannot increase it above the underlying supply realization. In this more general environment we assume that the distribution of underlying supply is exogenously given by F with a compact support.[52] Our proof then shows that the revenue maximizing-supply reduction by the seller reduces supply to $Q^{\star}(R)$ whenever the exogenous supply is higher than $Q^{\star}(R)$, and otherwise leaves the supply unchanged.

More interestingly our analysis lends itself to showing that the seller would like to fully reveal the realized supply: the seller thus finds transparency optimal both in the sense of setting a deterministic supply (or supply cap) and in the sense of revealing the seller's information about supply. To formalize this full revelation insight we enrich the base model of the paper as follows. We assume that the distribution of supply is exogenously given and commonly known. Before learning the realization of supply, the seller can publicly commit to an auction design (reserve price and supply restriction) and a revelation policy; a revelation policy maps the realization of supply to a distribution of public announcements (messages) from an arbitrary space of messages.[53] After committing to a revelation policy and an auction design, the seller learns the realization of supply and publicly announces the

[50] This argument hinges on re-assigning the revenue across supply realizations; in particular, the actual revenue conditional on a supply realization is not necessarily bounded by the reveue the seller would obtain by setting the deterministic supply fixed at the conditioning supply realization.

[51] In a working version of this paper, we provide a tighter bound on expected revenue, in which we average $\pi^{m}\left(s,Q\right)$ over the auxiliary distribution $J \equiv 1-(1-F)^{(n-1)/n}$. The bound presented in equation (3) is simpler to derive, and is sufficient for all our results.

[52] We can replace the assumption that the support of F is compact with other assumptions that guarantee that the optimal solution exists, such as for instance that there is a finite $q>0$ such that for all s, $v(q;s)=0$.

[53] We maintain the global assumption of this section that all induced auction equilibria have pure strategies.

message prescribed by the revelation policy. Then, the bidders learn their value and bid in the auction.

Theorem 6. [Optimality of Information Revelation] *The seller's expected revenue is maximized when the seller commits to fully reveal the realization of supply.*

Before presenting a surprisingly simple argument deriving this theorem from our preceding results, let us observe that Theorem 6 remains valid even if the seller does *not* optimize the reserve price and supply cap in the auction and these parameters of the auction are arbitrarily set, with no change in the proof. In addition, because we prove Theorem 6 for the environment in which the seller can commit to a revelation strategy, the same full revelation insights a fortiori holds true for environments where the seller cannot commit.

Proof. Suppose that the seller commits to a revelation strategy and this strategy leads to a message that induces the bidders to believe that the (conditional) distribution of supply is $\hat{F}$ with upper bound of support $\hat{Q}$. The revenue bound obtained in the proof of Theorem (5) gives

$$\mathbb{E}\left[\pi\right] \leq \int_0^{\hat{Q}} \mathbb{E}_s\left[\pi^m\left(s,x\right)\right] d\hat{F}\left(x\right),$$

and thus expected revenue is bounded above by the expected revenue obtained by the seller fully revealing to the bidders the realization of supply. In consequence, the seller's expected revenue is maximized when the seller ex ante commits to fully reveal the realization of supply. □

4.2 Separability of Optimal Supply and Reserve Price

The transparency result of the previous subsection substantially simplifies the seller's optimization problem. With reserve price R and deterministic supply Q—recall that optimal supply is deterministic—the revenue is

$$\begin{aligned}\mathbb{E}_s\left[\pi\right] = &\Pr\left(v\left(\frac{Q}{n};s\right) \geq R\right) \mathbb{E}\left[v\left(\frac{Q}{n};s\right)|v\left(\frac{Q}{n};s\right) \geq R\right] Q \\ &+ \Pr\left(v\left(\frac{Q}{n};s\right) < R\right) R\mathbb{E}\left[nv^{-1}\left(R;s\right)|v\left(\frac{Q}{n};s\right) < R\right].\end{aligned}$$

Because in Theorem 2 we proved that equilibrium strategies are symmetric, revenue depends on whether the marginal value for the per-capita quantity available, Q/n, is above or below the reserve price R.[54]

[54]Both reserve price and quantity restriction play a role in optimizing pay-as-bid auctions, except in the special case of complete information case which lends itself to some simplifications in the design analysis; we

To simplify the exposition, let us now assume that the signals s come from an atomless distribution on a subset of $\mathbb{R}$ and that the bidders' marginal values are increasing in the signal.[55]

Theorem 7. [Separable Optimization] *Let $R^{\star}$ be an optimal reserve, $Q^{\star}$ be the optimal supply in a pay-as-bid auction, and $\hat{s} = \inf\{s : v(Q^{\star}/n; \hat{s}) \geq R^{\star}\}$. Then, $\hat{s} \in \mathbb{R}$ and*

$$R^{\star} \in \arg\max_{R} R\mathbb{E}\left[v^{-1}(R; s)\middle| s < \hat{s}\right], \quad Q^{\star} \in \arg\max_{Q} Q\mathbb{E}\left[v(Q; s)\middle| s \geq \hat{s}\right].$$

To illustrate the usefulness of the separability that this theorem establishes, consider the following.

Example 1. Take some constants $\rho, \underline{s}, \overline{s} > 0$, such that $\overline{s} > \underline{s} \geq \rho\overline{Q}/n$ and suppose that s is distributed uniformly on $(\underline{s}, \overline{s})$ and $v(q; s) = s - \rho q$ for some constant $\rho > 0$. Thus, $\varphi(R; s) = (s - R)/\rho$. For every relevant deterministic supply Q and reserve price R is then the unique cut-off $\tau = \tau(Q, R) = R + \rho Q/n$ such that[56]

$$R = v\left(\frac{Q}{n}; \tau\right) = \tau - \rho\frac{Q}{n}.$$

For all $s < \tau(Q, R)$ the seller sells quantity $\varphi(R; s) = n(s - R)/\rho$ at price R; for all $s > \tau(Q, R)$ the seller sells quantity Q at price $v(Q/n; s) = s - \rho Q/n$. The seller's two-part maximization problem is then[57]

$$\max_{Q,R} \left(\frac{\overline{s} - \tau(Q, R)}{\overline{s} - \underline{s}}\right)\mathbb{E}_s\left[\left(s - \frac{\rho Q}{n}\right)Q|s > \tau(Q, R)\right] + \left(\frac{\tau(Q, R) - \underline{s}}{\overline{s} - \underline{s}}\right)\mathbb{E}_s\left[n\left(\frac{s - R}{\rho}\right)R|s < \tau(Q, R)\right].$$

The solution is

$$R^{\star} = \frac{\overline{s} + 3\underline{s}}{8}, \quad Q^{\star} = \left(\frac{3\overline{s} + \underline{s}}{8\rho}\right)n.$$

This solution gives the optimal deterministic supply of $Q^{\star}$ and the optimal reserve price of

discuss them in Appendix B and show there that under complete information optimizing just one of these instruments is sufficient.

[55]The insight of the separability theorem does not depend on this assumption; see Appendix H.2 for a general statement of the separability result.

[56]Note that since signals are uni-dimensional and values are strictly monotone in signal, the sets $\underline{\mathcal{S}}(Q, R)$ and $\overline{\mathcal{S}}(Q, R)$ are uniquely identified with such a cut-off τ.

[57]Since the uniform distribution is massless, we can ignore the event $s = s^{*}(Q, R)$. Also, for expositional purposes we constrain attention to cases in which the seller's problem has an interior solution.

$R^\star$ provided $Q^\star \leq \overline{Q}$.[58] The reserve price is binding because $\overline{s} > \underline{s}$ implies that $R^\star > \frac{7\underline{s}-3\overline{s}}{8} = \underline{s} - \rho\frac{Q^\star}{n} = v\left(\frac{Q^\star}{n};\underline{s}\right)$.

The cutoff type is $\hat{s} = \frac{\overline{s}+\underline{s}}{2}$ and the expected revenue is $\frac{n}{2\rho}\left(\left(\frac{3\overline{s}+\underline{s}}{8}\right)^2 + \left(\frac{\overline{s}+3\underline{s}}{8}\right)^2\right)$, which we can express in terms of the mean $m = \frac{\overline{s}+\underline{s}}{2}$ and the variance $V = \frac{(\overline{s}-\underline{s})^2}{12}$ of the signal distribution:

$$\text{Expected Revenue} = \frac{n}{2\rho}\left(\frac{m^2}{2} + \frac{3V}{8}\right).$$

The expected revenue is proportional to the number of bidders, a somewhat surprising consequence of the linearity of the problem. The expected revenue is also increasing in the mean and variance of the signal distribution and decreasing in the steepness ρ of the marginal value function. The monotonicity in the variance of the distribution means that a mean-preserving spread induces gains on high types that outweigh the losses on low types even when the seller doesn't know the types; the seller benefits from the upside while being able to limit the downside by setting the reserve price.

The separability of Theorem 7 allows us to compare optimally designed pay-as-bid reserve and supply to choices of a seller who sets the price (without optimizing over supply) and to the seller sets the supply allowing the price to be determined by Cournot-like market forces. In the context of the above example, the optimal price is half of the mean valuation for the initial unit, $p^{\text{MONOP}} = \frac{m}{2}$, and the optimal supply is $q^{\text{MONOP}} = \frac{m}{2\rho}n$ (that is the mean type utility on the optimal per-bidder supply is half of the utility on the initial unit), and hence

$$p^{\text{MONOP}} > R^\star \quad \text{and} \quad q^{\text{MONOP}} < Q^\star.$$

That is, the optimally designed pay-as-bid auction allocates a higher quantity at a lower (reserve) price than the classical monopolist's problem. This feature arises from the ability of the pay-as-bid seller to hedge the two design parameters against one another. In Appendix K, we establish these comparisons more generally, thus showing how Theorem 7 contributes to the literature on whether price or quantity is a better instrument for achieving desired market outcomes.[59]

The separability of the pay-as-bid designer's problem shown in Theorem 7 sharply contrasts with problem faced by a designer of a uniform-price auction, which we study in Section 5. In the uniform-price auction equilibrium bids are not (in general) unique, the monotonic-

[58] If $Q^\star > \overline{Q}$ then $Q = \overline{Q}$ is the optimal supply and the optimal reserve price $R = \frac{\rho\overline{Q}}{3n} + \frac{\underline{s}}{3}$ is given by the first order condition.

[59] Weitzman [1974] obtains conditions under which price or quantity regulation is preferred under stochastic demand and supply; Roberts and Spence [1976] find that a three-part system involving permits, penalties, and repurchase is preferable to any single-instrument system.

ity of bid in value cannot be assured, and it is not necessarily the case that uniform-price auctions can be optimized in a separable manner.

4.3 Comparative Statics

As an illustration of the separability theorem, consider the issue whether the seller benefits from a mean-preserving spread of the distribution of bidders' signal. We have seen that this is the case in Example 1 and we now extend the insight of this example to general signal distributions.

Proposition 1. *If the true marginal demand is increasing in the signal and linear and additively separable in quantity and the signal, then the seller's revenue in optimally designed pay-as-bid auction is increased by any mean-preserving spread of the distribution of bidders' signals.*

The assumptions of the proposition allow us to linearly renormalise the signal so as to make it one-dimensional and represent the marginal revenue as $v(q;s) = s - \rho q$. We conduct the proof under the assumption that the distribution σ has no atoms; because any distribution σ can be approximated via atomless distributions and the seller maximization is continuous with respect to such approximations, imposing this assumption is without loss of generality. Let $\pi(s)$ be the equilibrium revenue associated with signal s and notice that Theorem 7 implies that π is differentiable in s except possibly at $s = \hat{s}$:

- For $s < \hat{s}$, we have

$$\frac{d\pi}{ds} = \frac{d}{ds}\left[R^\star v^{-1}\left(R^\star; s\right)\right] = R^\star \frac{d}{ds}\left[\frac{1}{\rho}\left(s - R^\star\right)\right] = \frac{R^\star}{\rho}.$$

- For $s > \hat{s}$, we have

$$\frac{d\pi}{ds} = \frac{d}{ds}\left[Q^\star v\left(Q^\star; s\right)\right] = Q^\star \frac{d}{ds}\left[s - \rho Q^\star\right] = Q^\star.$$

Thus, $\frac{d\pi}{ds}$ is piecewise constant and its value for $s < \hat{s}$ is strictly below its value for $s > \hat{s}$ because $\frac{R^\star}{\rho} < Q^\star$; the latter inequality being satisfied because Theorem 7 gives us

$$R^\star = \frac{1}{2}\mathbb{E}\left[s | s < \hat{s}\right],$$
$$Q^\star = \frac{1}{2\rho}\mathbb{E}\left[s | s > \hat{s}\right]$$

(where the strict inequality follows from σ being atomless). Furthermore, at $s = \hat{s}$, $\frac{d\pi}{ds}$ has side derivatives as above. Defining $\frac{d\pi}{ds}|_{s=\hat{s}}$ to be a value weakly between the side derivatives, we obtain $\frac{d\pi}{ds}$ that is convex.

Now, consider an alternate signal distribution σ'. As with σ we can assume that this distribution is atomless.[60] The above analysis implies that

$$\mathbb{E}[\pi] = \int_{\underline{s}}^{\overline{s}} \pi(s)\, d\sigma(s) = \pi(\underline{s}) + \int_{\underline{s}}^{\overline{s}} \frac{d\pi}{ds}(1-\sigma(s))\, ds,$$

where $\underline{s}, \overline{s}$ are such that $\operatorname{Supp}\sigma, \operatorname{Supp}\sigma' \subseteq [\underline{s}, \overline{s}]$; note that the value we set for $\frac{d\pi}{ds}|_{s=\hat{s}}$ doesn't matter because of the assumption that σ is atomless. The optimal revenue under distribution σ' is bounded below by the revenue obtained with the reserve $R^\star$ and quantity $Q^\star$ that are optimal for distribution σ, and thus the difference in optimal revenues is at least

$$\begin{aligned}
\mathbb{E}_{s\sim\sigma'}[\pi(s)] - \mathbb{E}_{s\sim\sigma}[\pi(s)] &= \int_{\underline{s}}^{\overline{s}} \mu\pi(s)(\sigma(s)-\sigma'(s))\, ds \\
&= \int_{\underline{s}}^{\overline{s}} \frac{R^\star}{\rho}(\sigma(s)-\sigma'(s))\, ds + \int_{\hat{s}}^{\overline{s}} \left(Q^\star - \frac{R^\star}{\rho}\right)(\sigma(s)-\sigma'(s))\, ds \\
&= \frac{R^\star}{\rho}\left(\mathbb{E}_{s\sim\sigma}[s] - \mathbb{E}_{s\sim\sigma'}[s]\right) + \left(Q^\star - \frac{R^\star}{\rho}\right)\left(\int_{\hat{s}}^{\overline{s}} \sigma(s) - \sigma'(s)\, ds\right).
\end{aligned}$$

If σ' is a mean-preserving spread of σ, the left-hand term is (definitionally) zero. Because, as noted above, $Q^\star > R^\star/\rho$, we a mean-preserving spread σ' improves revenue if (but not necessarily only if)

$$\int_{\hat{s}}^{\overline{s}} \sigma(s) - \sigma'(s)\, ds > 0.$$

The latter condition is always satisfied when σ' is a mean-preserving spread of σ. QED

5 The Auction Design Game: Pay as Bid vs. Uniform Price

Sellers of homogenous goods are not restricted to running pay-as-bid auctions. The uniform-price auction is the other of the two most-commonly implemented format of auctions of homogenous goods. From a practical perspective, which of these two formats is preferred is

[60] As later we will assume that σ' is a mean-preserving spread of σ, it is also important that we can approximate such σ and σ' by sequences of atomless distributions σ_k and σ'_k that converge to σ and σ', respectively, and that are such that σ'_k is a mean-preserving spread of σ_k. To find such sequences, we can, for instance, set σ_k to be $\sigma + U_{\frac{1}{k}}$ and σ'_k to be $\sigma' + U_{\frac{1}{k}}$ where $U_{\frac{1}{k}}$ is the uniform distribution on $\left[-\frac{1}{k}, \frac{1}{k}\right]$.

a an important question that has be extensively studied both in the theoretical and empirical literature on divisible good auctions (see the introduction).[61] Unlike this literature—which compares the formats without taking the seller's endogenous choices into account—we explicitly model the seller's choice between the pay-as-bid and uniform-price formats (as well as among supply distributions and reserve prices) as an extensive-form game.

The auction design games have two stages. In the first stage, the seller commits to a reserve price, a distribution of supply, and the auction format (pay-as-bid or uniform-price); we also consider "subgames" in which the auction format is fixed. In the second stage, bidders participate in the specified auction. We consider perfect Bayesian equilibria of these games. This structure allows us to compare outcomes of optimally designed pay-as-bid and uniform-price auctions, and to discuss the economic implications of mechanism selection.

5.1 Revenue Superiority of Pay-as-Bid

We start our analysis of revenue-maximizing design with the case of uninformed seller and symmetrically informed bidders; in Section 6 we relax this assumption, allowing for asymmetrcially informed bidders. For the pay-as-bid auction, Theorem 1 states that equilibrium bids are essentially unique conditional on the distribution of supply, and Theorem 5 states that optimal supply is deterministic. Together these immediately imply that equilibrium revenue is unique in the pay-as-bid design game.

Corollary 5. [Essentially Unique Equilibrium in Pay-as-Bid Design Game] *In the pay-as-bid design game with symmetrcially informed bidders, the equilibrium revenue is unique. The revenue is*

$$\max_{R,Q} \Pr\left(v\left(\frac{1}{n}Q;s\right) \geq R\right) \mathbb{E}\left[v\left(\frac{1}{n}Q;s\right)\middle| v\left(\frac{1}{n}Q;s\right) \geq R\right] Q$$
$$+ \Pr\left(v\left(\frac{1}{n}Q;s\right) < R\right) \mathbb{E}\left[nv^{-1}\left(R;s\right)\middle| v\left(\frac{1}{n}Q;s\right) < R\right] R.$$

As noted in the discussion preceding Theorem 1, in the pay-as-bid auction equilibrium outcomes are unique but bids for never-realized quantities may not be uniquely defined. The equilibrium is thus essentially unique, and with slight abuse of terminology we also refer to

[61] From a theoretical perspective, we might be also interested in the question what general selling mechanism is optimal but in the environments we focus on, the bidders' private information is correlated and hence the seller can nearly extract their full surplus using Cremer-McLean-type mechanisms [Crémer and McLean, 1988]. In the benchmark case of the current section, in which bidders' information is symmetric, the full extraction of bidders' surplus is possible: e.g. the seller can ask all bidders to report their private information and set each bidder's allocation and payment in a way that fully extract the surplus of that among announced types that maximizes the seller's revenue.

it as the unique equilibrium of the pay-as-bid design game, ignoring potential multiplicity for infeasible quantities that have no impact on the observed outcomes. In this unique equilibrium bids are flat, and equal to the maximum of the reserve price R and the marginal value for the per-capita maximum $v(Q/n; s)$.

In the uniform-price design game the analysis is more complicated. With symmetrically informed bidders, equilibrium bids in the uniform-price auction are optimal for every realization of supply, a point first made by Klemperer and Meyer [1989]: the reason is that given the bids of others, every realization of supply determines for a bidder a residual supply curve and the bid of this bidder effectively serves to select the price-quantity pair from this residual supply curve; this choice does not depend on choices at other realization of supply as long as the resulting bid curve is downward sloping. In effect, two supply distributions with the same support admit the same set of equilibria, and if one supply distribution has a smaller support than another, its set of equilibrium bids is a weak superset of the other. This implies that the revenue maximizing equilibrium for deterministic supply is also revenue maximizing among all possible supply distributions. In this sense, deterministic supply is also optimal in uniform-price auctions.

Theorem 8. [Deterministic Dominance in Uniform-Price Design Game] *With symmetrcially informed bidders, for any equilibrium of the uniform-price design game* $((R, F), b)$*, there is a deterministic-supply equilibrium* $((R^\star, F^\star), b^\star(\cdot; s, R^\star, F^\star))$ *that generates weakly higher seller revenue. In this deterministic supply equilibrium bidders' bid are the same as in* $((R, F), b(\cdot; s, R, F))$

$$b^\star(\cdot; s, R^\star, F^\star) = b(\cdot; s, R, F),$$

and the deterministic supply and reserve are given by:

$$\begin{aligned}(R^\star, Q^\star) \in \arg\max_{R,Q} \Pr\left(b^\star\left(\frac{1}{n}Q; s\right) \geq R\right) \mathbb{E}\left[b^\star\left(\frac{1}{n}Q; s\right)\middle| b^\star\left(\frac{1}{n}Q; s\right) \geq R\right] Q \\ + \Pr\left(b^\star\left(\frac{1}{n}Q; s\right) < R\right) \mathbb{E}\left[n b^{\star-1}(R; s)\middle| b^\star\left(\frac{1}{n}; s\right) < R\right] R.\end{aligned}$$

In particular, the resulting cumulative distribution function of supply is $F^\star(Q) = 1[Q \geq Q^\star]$.

Note that the bid functions $b(\cdot; s, R, F)$ depend on the bidders' signal as well as the reserve prices R and supply distributions F chosen by the seller. When there is no risk of confusion, when referring to the bids on the equilibrium path we sometimes suppress the dependence on R and F.

Remark 1. Theorem 8 does not imply that all equilibria of the uniform-price design game have deterministic supply. Because bidders' strategies can depend on the revealed distribution of supply, it is possible that selecting a deterministic supply will yield lower revenue than random supply. Consider bidders who bid the reserve price when supply is deterministic, but submit relatively aggressive bids otherwise. Then the seller could concentrate the supply distribution around the deterministic optimum, but retain some randomness to ensure that bidders submit aggressive bids. This will revenue-dominate deterministic supply, where bidders submit relatively weak bids.

The equilibria of the uniform-price game generate weakly less seller revenue than the unique equilibrium of the pay-as-bid design game, hence the pay-as-bid design game yields greater revenue than the uniform-price design game in general.

Theorem 9. [Revenue Comparison of Design Games] *With symmetrcially informed bidders, the unique equilibrium of the pay-as-bid design game generates weakly greater revenue than all equilibria of the uniform-price design game, and there is an equilibrium of the uniform-price design game that generates the same expected revenue as the unique equilibrium of the pay-as-bid design game.*

The revenue comparison is strict for all uniform-price equilibria in which bids $b^{\star}(\cdot; s, R^{\star}, F^{\star})$ are strictly below the realized marginal value $v(Q/n; s)$. Such equilibria are typical in the sense that in the uniform-price auction, for any Q and s, any price $p \in [R^{\star}, v(Q/n; s)]$ is supportable in equilibrium. The proof of Theorem 9 leverages the optimality of deterministic supply established in Corollary 5 and Theorem 8 above. This major endogenous simplification allows us to show that for any signal s, the equilibrium market clearing price $p^{\star\text{PABA}}(s)$ in pay-as-bid design game is weakly higher than the equilibrium market clearing price $p^{\star\text{UPA}}(s)$ in any equilibrium of the uniform-price design game. The full proof is provided in the appendix.

Finally, consider the joint auction design game. In the auction design game the designer commits to implement a pay-as-bid or uniform-price auction, and then the selected auction format is run. Theorem 9 implies that in the auction design game, the seller either implements a pay-as-bid auction or bidders bid aggressively in the uniform-price design game.

Corollary 6. [Revenue Equivalence Across Perfect Bayesian Equilibria] *All equilibria of the auction design game are revenue equivalent. Furthermore, the seller either implements a pay-as-bid auction or is indifferent between the pay-as-bid and uniform-price auctions.*

5.2 Welfare Ambiguity

Theorem 9 states that revenue can be compared between the pay-as-bid and uniform-price design games, and that the pay-as-bid auction generates weakly greater revenue. As it turns out other outcomes such as optimal quantity sold, optimal reserve price, and the resulting bidder and total welfare are not generally comparable between the uniform-price and pay-as-bid design games. This follows from equilibrium nonuniqueness in the uniform-price design game. In equilibrium the seller can be rewarded with relatively high pay-as-bid bids given deterministic supply while lower (equilibrium) uniform-price bids are employed off-path. In Appendix I we provide a natural equilibrium selection in the uniform-price auction which provides strictly lower revenue than the optimal pay-as-bid auction. If in the dynamic game bidders employ such a low-revenue bid profile whenever a particular reserve price and quantity is *not* selected the designer can be induced to implement a particular quantity distribution and reserve price. This observation gives us:

Theorem 10. [Ambiguous Quantity and Reserve Comparison] *Let $Q^{\star PAB}$ and $R^{\star PAB}$ be optimal quantity and reserve in the pay-as-bid design game. There is $\varepsilon > 0$ such that for all $Q^\star, R^\star$ with $|Q^{\star PAB} - Q^\star| < \varepsilon$ and $|R^{\star PAB} - R^\star| < \varepsilon$, there is an equilibrium of the uniform-price design game in which the designer selects deterministic quantity $Q^\star$ and reserve $R^\star$.*

Proof. Consider the bidding strategy b, where $b(\cdot;\cdot,R,F)$ is the flat bid associated with the pay-as-bid auction if $R = R^\star$ and F is a degenerate distribution on $Q^\star$, and low uniform-price equilibrium bids otherwise. In the pay-as-bid auction the designer's revenue varies continuously in the reserve price R and quantity Q. Then since optimal pay-as-bid revenue is strictly above optimal uniform-price revenue with strictly lower bids, as long as $R^\star$ and $Q^\star$ are relatively near $R^{\star\text{PAB}}$ and $Q^{\star\text{PAB}}$ the designer in the uniform-price design game will implement deterministic supply $Q^\star$ and reserve $R^\star$. □

Corollary 7. [Ambiguous Bidder Welfare Comparison] *The uniform-price design game admits equilibria in which bidder surplus is higher and equilibria in which bidder surplus is lower than in the unique equilibrium of the pay-as-bid design game.*

Theorem 10 implies that there are equilibria of the uniform-price design game in which $R^{\star\text{UPA}} < R^{\star\text{PAB}}$ and $Q^{\star\text{UPA}} > Q^{\star\text{PAB}}$, as well as equilibria in which $R^{\star\text{UPA}} > R^{\star\text{PAB}}$ and $Q^{\star\text{UPA}} < Q^{\star\text{PAB}}$. The former generate higher bidder surplus than the pay-as-bid design game, and the latter generate lower bidder surplus than the pay-as-bid design game.[62]

The analysis of Theorem 10 further implies that there are equilibria of the uniform-price design that are worse for all market participants: revenue, bidder surplus, and aggregate

[62] [Ausubel et al., 2014] show that revenue and efficiency cannot be generically compared between the pay-as-bid and uniform-price auction formats, and that the comparison may vary with model specification. This is distinct from Corollary 7, which states that the comparison is ambiguous even when the model is known. Ambiguity in our model arises from equilibrium selection. Furthermore, their examples of ambiguity rely on ex-ante asymmetries between bidders, while ours do not.

surplus may all be lower in the uniform-price auction than in the unique equilibrium of the pay-as-bid design game.

Corollary 8. [Pay-as-bid Preferred By Seller ad Buyers] *The uniform-price design game admits equilibria in which both the seller's revenue and bidders' surplus is lower than in the unique equilibrium of the pay-as-bid design game.*

6 Asymmetric Information among Bidders

In this final section, we relax the assumption that bidders are symmetrically informed, and allow for heterogeneous signals s_i. Without loss of generality we assume that $s_i = (s, \varepsilon_i)$ where s is a common signal known to all bidders while ε_i is idiosyncratic and privately known only to bidder i; notice that we do not require that ε_i and ε_j are independent nor we require that they are identically distributed. This separation of signals into common and idiosyncratic components is inessential but simplifies the comparison to our base model with only a common signal. For the sake of expositional simplicity we normalize the signals so that each idiosyncratic signal ε_i can take value 0 and we treat the case of all idiosyncratic signals taking value 0 as the base common signal case. Let $\mathcal{S}_s = \operatorname{Supp} s$ and $\mathcal{S}_\varepsilon = \operatorname{Supp} \varepsilon$.

Marginal values are now a function of both the common and idiosyncratic signals, $v : \mathbb{R}_+ \times \mathcal{S}_s \times \mathcal{S}_\varepsilon \to \mathbb{R}_+$. We retain the assumption that $v(\cdot; s, \varepsilon)$ is nonnegative and strictly decreasing when strictly positive, and place no assumptions on how private information affects marginal values. We assume that bidder information has full support, so that $\operatorname{Supp} \varepsilon_i|_{s,\varepsilon_{-i}} = \mathcal{S}_\varepsilon$, but otherwise there are no distributional assumptions on s, ε_i, or their interrelation.

Our analysis of optimal design relies on the following key lemma proven in the appendix.

Lemma 1. [Incomplete Information Bound] *For any public signal s, the market clearing price for the maximum quantity $\overline{Q}$ in any equilibrium of pay-as-bid is bounded between the smallest and largest marginal value at the per-capita maximum quantity,*

$$\min_i \operatorname{ess\,inf}_\varepsilon v^i \left(\frac{1}{n} \overline{Q}; s, \varepsilon \right) \leq p\left(\overline{Q}; s, \varepsilon\right) \leq \max_i \operatorname{ess\,sup}_\varepsilon v^i \left(\frac{1}{n} \overline{Q}; s, \varepsilon \right).$$

In the sequel we focus on the case with only a little asymmetric information, which we model by assuming that the support of $v(q; s, \cdot)$ is small.

Definition 1. For $\delta > 0$, we say that informational asymmetry is δ-small if, for all $(s, \varepsilon) \in \mathcal{S}_s \times \mathcal{S}_\varepsilon$, $\sup_q |v(q; s, 0) - v(q; s, \varepsilon)| < \delta$.

For small informational asymmetry, we show that the expected revenue in any equilibrium of an optimal pay-as-bid auction—that is pay as bid with optimal reserve price and supply—with asymmetric private information is nearly above the expected revenue in the unique equilibrium of the optimized auction when bidders' information is symmetric:

Theorem 11. [Revenue Loss from Informational Asymmetry] *Suppose the informational asymmetry is δ-small. Then, the expected revenue in any equilibrium of the optimal pay-as-bid auction is bounded from below by the expected revenue in the unique equilibrium of the optimal pay-as-bid auction in the base symmetric information environment in which $\epsilon = 0$ decreased by δQ^*, where Q^* is the optimal supply in the symmetric information environment.*

In the next lemma, for shortness, we say that the expected revenue is *nearly bounded below* by a bound if the expected revenue is above the bound decreased by δQ^*, where Q^* is the optimal supply in the symmetric information environment. In Section 6.1, in the context of δ-small informational asymmetry, we say that the expected revenue is nearly bounded below by a bound if the expected revenue is above the bound decreased by $2\delta Q^*$, and we analogously understand *nearly above* and *nearly indifferent*.

Proof. Optimal revenue in a pay-as-bid auction with asymmetric information is bounded below by the expected revenue generated by any fixed supply Q and reserve R. Fixing supply and reserve at $(Q^*, R^* - \delta)$, the optimal supply and reserve in the symmetric information environment, Lemma 1 shows that the market clearing price is bounded below by $\underline{p} = \text{ess inf}_\zeta\, v(Q^*/n; \zeta) \geq \text{ess inf}_s\, v(Q^\star/n; (s, 0)) - \delta$,where the inequality follows from the informational asymmetry being δ-small. With symmetric information and optimal supply, bids are flat and equilibrium per-unit revenue conditional on signal s is $\max\{v(Q^*/n; (s, 0)), R^*\}$; with asymmetric information, bids may be decreasing, but equilibrium per-unit revenue conditional on signal s is bounded below by $\max\{\underline{p}, R^* - \delta\}$. Given reserve $R^* - \delta$ and a δ-small informational asymmetry, at least as many units are sold under asymmetric information as are sold under symmetric information with reserve R^*, and the result follows. □

Note that this theorem is not a simple limit result. First, in environments for which purification results have been proven, a limit of equilibria as we decrease the import of idiosyncratic signals is a mixed-strategy equilibrium in the limit environment but in Theorem 11 we bound the revenue from below by a pure strategy equilibrium in the limit environment. Second, there are so far no purification results for such infinitely-dimensional discontinuous games as pay-as-bid auctions. We are able to establish this theorem because of our earlier results that when bidders' information is symmetric then optimal supply is deterministic. In light of this transparency insight, Theorem 11 follows from the following.

Lemma 2. *In an optimal pay-as-bid auction with deterministic supply, the expected revenue in any equilibrium with asymmetric private information is nearly above the expected revenue in the unique equilibrium when bidders' information is symmetric.*

This lemma in turn follows from Lemma 1, in which we establish that the market clearing price is bounded below by the lowest marginal value $v(\cdot;\cdot,\cdot)$ of per capita supply. When $v(\cdot;\cdot,\cdot)$ is within δ of base $v(\cdot;\cdot,0)$, this lowest marginal value of per capita supply is weakly above $v\left(\frac{Q}{n};\cdot,0\right)-\delta$. By setting the deterministic quantity Q at optimal value at symmetic information and lowering the reserve price by δ with respect to optimal reserve R at symmetric information, the seller can sell at least the same quantity of the good. When bidders are asymmetrically informed, the per-unit price is bounded below by $\max\left(v\left(\frac{Q}{n};s,0\right)-\delta,R-\delta\right)$, while $\max\left(v\left(\frac{Q}{n};s,0\right),R\right)$ is the per-unit revenue in the unique equilibrium when bidders' information is symmetric and equals $(s,0)$. Thus Lemma 2 obtains.

6.1 Pay-as-Bid vs. Uniform-Price

We now show that the revenue comparison results of Section 5 continue to hold for asymmetrically informed bidders when the informational asymmetry is small. The comparison requires us to understand the behavior of asymetrically informed bidders in pay-as-bid auctions and uniform-price auctions, and the design response of the seller. The logic developed above, giving and approximate revenue bound in the pay-as-bid auction when the informational asymmetry is δ-small, also applies to the uniform-price auction, with the exception that equilibrium may be nonunique. In the uniform-price auction with private information, every equilibrium generates revenue that is weakly below quantity times the maximum marginal value for per capita supply.

Lemma 3. *Fix a reserve price and supply distribution. In a uniform-price auction, the expected revenue in an equilibrium with asymmetrically informed bidders is nearly below the expected revenue under truthful bidding with symmetrically informed bidders.*

Note that the truthful bidding revenue is an upper bound on the equilibrium revenue.

Proof. In the uniform-price auction, the parts of bids that set the price in equilibrium are bounded above by truthful reporting. Let Q be a realisation of supply; notice that Q is bounded above by the maximum supply $\bar{Q}$. Conditional on this revenue draw, for any fixed $\delta>0$, the expected revenue under asymmetric information that is within δ of $v(\cdot;s,0)$ is bounded above by $Q\left[v\left(\frac{Q}{n};s,0\right)+\delta\right]\leq Qv\left(\frac{Q}{n};s,0\right)+\delta\bar{Q}$. The result follows because $Qv\left(\frac{Q}{n};s,0\right)$ is the revenue conditional on the common signal being s under truthful bidding when bidders are symmetrically informed. □

In light of our earlier analysis, we obtain the following

Theorem 12. *With optimal reserve price and supply, the expected revenue in the pay-as-bid auction is nearly above expected revenue in the uniform-price auction.*

Theorem 13. *Given any deterministic supply Q and reserve price R, expected revenue in the pay-as-bid auction is nearly above expected revenue in the uniform-price auction. Moreover, the seller is nearly indifferent between any equilibrium of the pay-as-bid auction and any revenue-maximizing equilibrium of the uniform-price auction.*

Proof. The first statement follows from Lemmas 2 and 3. To prove the second statement, consider a uniform-price auction where bids, conditional on common signal s, are bounded below by $\underline{b}(s) = \max\{R, \text{ess inf}_{\zeta|s}\, v(Q/n;\zeta)\}$: bidding below $\underline{b}(s)$ cannot yield additional quantity, and by construction, when $b \geq \underline{b}(s)$ the marginal value for all units obtained is weakly positive. It follows that there is an equilibrium in which bids are at least $\underline{b}(s)$, and the second claim follows. □

Theorem 12 states that, while the uniform-price auction might generate greater revenue than a pay-as-bid auction, this difference will not be large without a significant informational asymmetry among bidders. A weakened version of Corollary 6 holds, in which the seller either strictly prefers a pay-as-bid auction, or is approximately indifferent between the pay-as-bid and uniform-price auctions.

6.2 Approximate Optimality of Transparency

Our analysis of elastic supply and mixed strategy equilibria in Appendix (A) shows that if buyers' values are regular, a deterministic supply curve maximizes the seller's revenue. In this subsection we apply this analysis to the design of optimal pay-as-bid auctions in the presence of small informational asymmetries.

Theorem 14. **[Essential optimality of deterministic supply]** *The optimal deterministic supply curve without asymmetric information is approximately optimal when asymmetric information is δ-small.*

The proof of Theorem (14) requires some technicalities. Its gist is as follows. For a sequence $\langle \delta_t \rangle_{t=1}^{\infty}$ decreasing to 0, let K^{δ_t} be an associated sequence of optimal random supply curves in models where informational asymmetries are δ_t-small. Variants of Helly's selection theorem (see, e.g., Reny [2011], Lemma A.10) imply that there is a limiting distribution $K^\star$ such that $K^{\delta_t} \to_t K^\star$ pointwise almost everywhere on a subsequence, on which we focus in the sequel. Holding fixed a bid profile, bidder utility is continuous in joint randomization over reserve and supply (equivalently, a random supply curve), and the existence result of Woodward [2019a] implies that the limit of bidding equilibria under distributions K^{δ_t} is an (potentially mixed-strategy) equilibrium under distribution $K^\star$, given no asymmetric information. Because Appendix (A) shows that deterministic elastic supply dominates any random residual supply curve, it follows that either $K^\star$ is an elastic supply curve, or is nonoptimal. If it is nonoptimal, the optimal deterministic supply curve raises more expected revenue than the random supply curve implied by K^{δ_t}, for some t, contradicting optimality.

6.3 Relationship to Empirical Findings

Our results provide a theoretical explanation for the popularity of the pay-as-bid format—cross-country comparisons find that pay-as-bid auctions are more than twice as prevalent as uniform-price auctions (see Brenner et al. [2009] for treasury securities; see Del Río [2017] for electricity auctions). If revenue-interested sellers are at worst indifferent or close to indifferent between pay-as-bid and uniform-price auctions, it is natural to suspect that pay-as-bid auction should be implemented more frequently.

Corollary 6 and Theorem 12 provide an explanation of the empirical finding that revenues in pay-as-bid are close to the counterfactual revenues in uniform-price (Kang and Puller [2008], Hortaçsu and McAdams [2010], Hortaçsu, Kastl, and Zhang [2018], and others). The explanation is two-fold. First, the auction format is selected by the seller and a revenue-maximizing seller weakly prefers the uniform price format only if this format is nearly equivalent to pay as bid. Second, the optimal pay-as-bid and uniform-price auctions generate the same revenue only in the seller-optimal equilibrium of the uniform-price auction and this is precisely the equilibrium in which bids are equal to marginal values. The latter equality is imposed in counterfactual revenue estimation of uniform-price auctions in the above mentioned studies, which assume truthful reporting in the uniform-price auction. Our results thus suggest that the empirical ambiguity of cross-mechanism revenue comparison is strongly tied to sellers' endogenously selecting the auction format and to the empirical literature equilibrium selection assumption.[63]

7 Conclusion

We have studied multi-unit auctions in an environment in which there is only limited asymmetry of information between bidders, but the seller (or auction designer) is potentially much less informed. For the limit case in which bidders' information is symmetric, we have established a mild condition for equilibrium existence as well as established equilibrium uniqueness and provided a tractable representation of bids.[64] We also proved that the limit equilibrium,

[63] While our revenue equivalence benchmark is new, the presence of low-price equilibria in Uniform Price designs have long been recognized; see Introduction.

[64] We hope that the tractability of our representation will stimulate future work on this important auction format. Wittwer [2017] discusses the intuition behind our representation.

Paper	**Data**	**Method**	σ/μ	**Conclusion**
FPV (2002)	France	PABA → CF UPA	1.27%	PABA > UPA
AS (2006)	France	PABA → CF UPA	3.78%	UPA > PABA
Umlauf (1993)	Mexico	Natural experiment	11.16%	UPA > PABA

Table 1: Revenue comparisons between auction formats, in comparison to the standard deviation of noncompetitive demand scaled by mean aggregate supply (Q); "CF" is "counterfactual."

without informational asymmetries among bidders, provides a lower bound on revenues in the presence of informational asymmetries.

We have used these results to analyze the design problem of the seller, allowing for bidders' private information. In particular, we established that revenue-maximizing pay-as-bid auctions generate more revenue than uniform-price auctions, and strictly more revenue in most cases, but welfare comparisons are inherently ambiguous. In particular, it is possible that revenue-maximizing pay-as-bid auctions are not only revenue—but also welfare—superior to uniform-price auctions.

As part of our analysis we established revenue equivalence between revenue-maximizing pay-as-bid auctions and the revenue-maximizing equilibram of uniform-price auctions. Our revenue equivalence benchmark provides an explanation for the empirical findings that find approximate revenue equivalence between the two formats by imposing—for identification and counterfactual analysis—that the revenue maximizing equilibrium obtains in uniform-price auctions; this is precisely the assumption that we show leads to theoretical revenue equivalence. Furthermore, although we describe our revenue equivalence results in terms of optimized auctions, the equivalence of pay-as-bid and seller-optimal uniform-price auctions depends only on deterministic supply, which may be either a consequence of optimality or exogenously given.

Our revenue equivalence result also provides a plausible explanation for the second-order details of empirical findings regarding multi-unit auction revenue; Table 1 relates revenue comparisons from the literature to normalized randomness in aggregate supply, and suggests that increased uncertainty in supply improves the relative performance of the uniform-price auction over the pay-as-bid auction.[65] This is in line with our result that optimal supply is deterministic, and that with deterministic supply—independent of whether the mechanism is optimal—all equilibria of the uniform-price auction are revenue-dominated by the unique

[65]The small number of results summarized in Table 1 compared to the larger number of results in the introduction is a matter of data availability. The papers in Table 1 separately summarize aggregate and noncompetitive supply and provide the first two moments, allowing a simple calculation of the relative randomness of a single run of an auction.

equilibrium of the pay-as-bid auction.

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A Elastic Supply and Mixed-Strategy Equilibria

In the main text we (mostly) focus on designing a supply distribution, potentially subject to a simple reserve price. However, our essential insights remain valid if we allow the seller to design a potentially stochastic elastic supply curve. We now relax the assumptions that the seller separately sets the reserve price and supply, and analyze pay-as-bid with optimally designed elastic supply. [66] In what follows, we consider a bid function $b(\cdot; s)$ which depends on bidder-common information s. To simplify notation, we omit this signal from the bid function (and other functions) where it is not of importance.

[66]Note that in the special case of our model with perfectly correlated types, $(s, \varepsilon_i) = (s, 0)$, the seller can fully extract the bidders' rents. Furthermore, in our model of asymmetrically informed bidders with demand curves coming from δ support around commonly known (among bidders) demand, we get within order-δ of full extraction: even if idiosyncratic information is i.i.d., the seller can extract from the bidders the common part of the demand curve.

Our first step is to convert stochastic elastic supply to a random reserve price. Consider the seller who selects a distribution over reserve prices, possibly correlated with the distribution of quantity. Let $K(Q;R)$ be a *supply-reserve distribution*, giving the probability that realized quantity $\tilde{Q} \leq Q$ or realized reserve price $\tilde{R} > R$,

$$K\left(Q;R\right) = \Pr\left(\tilde{Q} \leq Q\right) + \Pr\left(\tilde{Q} > Q, \tilde{R} > R\right).$$

For the moment, we assume that K is massless.[67] Note that conditional on aggregate demand $p(\cdot)$, $K(Q;p(Q))$ is the probability that realized aggregate supply is below Q: either realized supply $\tilde{Q} \leq Q$, or realized reserve $\tilde{R} > p(Q)$ and quantity is constrained.

Holding the supply-reserve distribution K fixed, fix a bidder i and consider the aggregate demand of her opponents. Allowing for mixed strategies, bidder i faces residual supply $Q(\cdot;\xi)$, where ξ indexes the joint distribution of her opponents' potentially mixed strategies. Bidders' ex post utility is determined by realized quantity and payment, and thus the basic form of interim utility is unaffected by the introduction of a random reserve price or the possibility of mixed strategies. Then the basic form of a bidder's first order condition is unchanged from the analysis in Lemma 12, and random reserve affects only the distribution of realized quantity. In the language of Lemma 12,

$$\begin{aligned} G^i\left(q;b\right) &= \mathbb{E}_\xi\left[K\left(q+Q\left(b;\xi\right);b\right)\right], \\ \text{and } G^i_b\left(q;b\right) &= \mathbb{E}_\xi\left[K_Q\left(q+Q\left(b;\xi\right);b\right)Q_p\left(b;\xi\right) + K_R\left(q+Q\left(b;\xi\right);b\right)\right]. \end{aligned} \tag{4}$$

When the reserve price is fixed, $K_R = 0$ and (4) is identical to what we find in equation (11). Alternatively, the seller might implement a random reserve with unconstrained supply, $K_Q = 0$. In this latter case, the bidder's pointwise first order condition is

$$\left(v\left(q\right) - b\left(q\right)\right)\mathbb{E}_\xi\left[K_R\left(q+Q\left(b\left(q\right);\xi\right);b\right)\right] = \mathbb{E}_\xi\left[K\left(q+Q\left(b\left(q\right);\xi\right);b\right)\right].$$

Since $K_Q = 0$ implies that K is independent of q (and thus Q is independent of ξ), we write this in terms of only the distribution of reserve prices F^R,

$$\left(v\left(\varphi\left(p\right)\right) - p\right)F^R_p\left(p\right) = F^R\left(p\right).$$

Facing the reserve distribution F^R, an optimal bid (i) does not depend on opponent bids, and (ii) is either pointwise optimal, or the first order conditions are ironed (c.f. Woodward-2014)

[67] In general, $K(Q;R) = 1 - \Pr(\tilde{Q} > Q, \tilde{R} < R)$, and K is not a cumulative distribution function. In the absence of mass points, however, $\Pr(\tilde{Q} \leq Q, \tilde{R} \leq R) = K(Q;R) - K(0;R)$, and the cumulative distribution function is in one-to-one correspondence with K.

over an interval. For the results below, we consider first-order optimal bids.

Definition 2. Given a distribution of reserve prices F^R, b is *first-order optimal* with respect to F^R if:

1. b is constant only if $b(q)$ is a mass point of F^R;
2. Wherever b is strictly decreasing, it solves the pointwise first order condition: $(v(q) - b(q))F_p^R(b(q)) = F^R(b(q))$;
3. Wherever b is constant, it solves the ironed first order condition:

$$\left(F^R(b(q)) - F^R(\underline{p})\right)\left(v(\overline{\varphi}(p)) - \underline{p}\right) = \left(b(q) - \underline{p}\right) F^R\left(\underline{p}\right), \text{ where } \underline{p} = \lim_{q' \searrow \overline{\varphi}(p)} b(q').$$

A bid function which is first-order optimal satisfies pointwise first order conditions where applicable, and ironing conditions elsewhere, but may not be globally optimal. Intuitively, the ironing conditions state that the marginal gain from slightly extending the constant interval (marginal additional quantity with probability $F^R(b(q)) - F^R(\underline{p})$) must equal the marginal cost from the same (marginal additional payment with probabiliy $F^R(\underline{p})$). First-order optimal bids are an auxiliary tool to show optimality of deterministic supply functions. Our analysis of revenue properties assumes only that the first order conditions are satisfied, not that bids are optimal. However, any (globally) optimal bid function must satisfy the first-order optimality conditions; since our analysis ultimately shows that any first-order optimal bid in a discriminatory auction generates lower revenue than a uniform-price auction with a random reserve price, full optimality is not essential.

Let $G^K(\cdot; b, Q)$ be the distribution of realized quantity given stochastic-elastic supply K, bid function b, and (potentially random) residual supply Q, and let $G^R(\cdot; b)$ be the distribution of realized quantity given reserve distribution F^R and bid function b. As mentioned above, G^R does not depend on Q because, under random reserve, supply does not depend on opponent bids. Letting ξ represent randomness in residual supply (e.g., mixed strategies for a bidder's opponents)[68] we have

$$\begin{aligned}
G^R(q;b) &= 1 - F^R(b(q)), \\
\frac{d}{dq}G^R(q;b) &= -F_p^R(b(q))\,b_q(q); \\
G^K(q;b,Q) &= \mathbb{E}_\xi\left[K(q+Q(b(q);\xi),b(q))\right], \\
\frac{d}{db}G^K(q;b,Q) &= \mathbb{E}_\xi\left[K_q(q+Q(b(q);\xi))\,Q_p(b(q);\xi) + K_p(q+Q(b(q);\xi),b(q))\right], \\
\frac{d}{dq}G^K(q;b,Q) &= \frac{d}{db}G^K(q;b,Q) + \mathbb{E}_\xi\left[K_q(q+Q(b(q);\xi))\right]. \qquad (5)
\end{aligned}$$

The expected revenue generated by bid b under realized quantity distribution $G^{\cdot}$ is[69]

$$\pi(b;G) = \int_0^Q \int_0^q b(x)\,dx\,dG^{\cdot}(q).$$

We now show that deterministic elastic supply generates higher revenue than any random supply. We begin with a bid function b which is a best response to residual supply distribution $G^i(\cdot;b)$. If this bid function is nowhere-flat, we show that there is a reserve distribution F^R such that b is first-order optimal; we show that the realized distribution of quantity under random reserve first order stochastic dominates the residual supply distribution $G^i(\cdot;b)$. Otherwise, if the bid function b contains a flat interval, there is a mass point in the realized price distribution. We assign this mass in the realized price distribution to the same price in an exogenous reserve distribution. Under random reserve, this creates an incentive to extend the flat-bid interval beyond the flat interval in the original bid b; by definition, the first-order optimal response to F^R, b^R, cannot equal b, but we construct F^R so that $b^R \geq b$, and $b^R = b$ wherever the former is strictly decreasing. We then show that we can arbitrarily approximate the first-order optimal bid b^R to F^R with a *strictly* decreasing bid function $\tilde{b}^R$, associated with some random reserve distribution $\tilde{F}^R$, and that the distribution of realized quantity under this approximation approximates the distribution of quantity under b^R. Then since $b^R \geq b$ and $\tilde{b}^R \approx b^R$, it follows that $\tilde{b}^R$ (at least) arbitrarily approximates the revenue generated by b. Since we show that any strictly decreasing first-order optimal bid $\tilde{b}^R$generates strictly less revenue than some uniform-price auction (Theorem 15), which is in turn weakly dominated by the unique equilibrium of a pay-as-bid auction, it follows that equilibria in pay-

[68] In the main text we focus on pure strategies. In this analysis we allow for mixed strategies, allowing us to show that all randomness — exogenous or otherwise — is detrimental to the seller's revenue.

[69] This is the revenue raised by a single bidder; aggregate revenue is the sum over all bidders. Our analysis below shows that randomness decreases the expected transfer the seller expects from a single bidder. It is immediate that the seller's aggregate revenue is maximized by a deterministic mechanism.

as-bid auctions with stochastic-elastic supply yield lower expected revenue than equilibria in pay-as-bid auctions with deterministic-elastic supply.

Note that, although we separately specify the supply-reserve distribution K and the mixed strategy index ξ, the proofs are agnostic to the source of randomness in a bidder's residual supply. Therefore these arguments also show that mixed strategy equilibria raise lower revenue than the unique deterministic equilibrium of a pay-as-bid auction with deterministic elastic supply.

Lemma 4. *Let b be a best response bid curve under residual supply distribution G^i, generated by supply-reserve distribution K and stochastic aggregate demand Q. There is a reserve distribution F^R with naive best response b^R such that $\pi\left(b^R; G^R\right) \geq \pi\left(b; G^i\right)$.*

Proof. First, recall that under a random reserve price the agent's pointwise first order conditions are

$$(v\left(q\right) - b\left(q\right)) F_p^R\left(b\left(q\right)\right) = F\left(b\left(q\right)\right).$$

It is straightforward to show that, over any interval on which $b^R(\cdot)$ is strictly decreasing,

$$\frac{d}{db} \ln F^R\left(b\right) = \left(v\left(\varphi\left(b\right)\right) - b\right)^{-1}.$$

Hence, given any initial condition $F^R(p)$ and any price $p' < p$, we have

$$F^R\left(p'\right) = \exp\left(-\int_{p'}^{p} \frac{1}{v\left(\varphi\left(x\right)\right) - x} dx\right) F^R\left(p\right). \tag{6}$$

That is, over any interval on which b is strictly decreasing, there is a reserve distribution F^R for which b is first-order optimal.

We now construct a reserve distribution F^R with a first-order optimal bid b^R that yields weakly greater expected revenue than the best response bid b under residual supply distribution G^i. To start, define an auxiliary function $X : [\underline{b}, \overline{b}] \to [0,1]$ with $X(b(0)) = 1$, where $\overline{b} = b(0)$ and $\underline{b} = \inf\{b(q)\colon G^i(q) < 1\}$, and for any (maximal) interval (q_ℓ, q_r) on which b is constant,[70] let $X(b(q_r)) = 1 - G^i(q_\ell)$. Over any (maximal) interval $[q_\ell, q_r]$ on which b is strictly decreasing, define $X(b(q))$ by equation 6 with initial condition $X(b(q_\ell)) = 1 - G^i(q_\ell)$. Note that X may contain mass points (where b is flat), and that these mass points may be inconsistent; i.e., it may be that $\lim_{p' \searrow p} X(p') > X(p) > \lim_{p' \nearrow p} X(p')$.[71] Let D represent the set of discontinuities of X, and for any $p \in D$ let $\lambda(p) = \lim_{p' \searrow p} X(p')/X(p)$. For any

[70] The interval (q_ℓ, q_r) is maximal (in this respect) if there is no $\varepsilon > 0$ such that b is constant on $(q_\ell - \varepsilon, q_r)$ or $(q_\ell, q_r + \varepsilon)$.

[71] We show below that the opposite inequality cannot hold.

$p \in [\underline{b}, \overline{b}]$, let $F^R(p) = X(p) \prod_{p' \in D, p' \leq p} \lambda(p')$. By construction, F^R is a distribution. Since equation 6 is scale-invariant, wherever b is strictly decreasing it is first-order optimal with respect to F^R.

Now, consider the distributions of realized quantity given bid b^R under reserve distribution F^R, and given the best response bid b under residual supply distribution G^i. Suppose that q_ℓ, q_r are such that $q_\ell < q_r$, $G^R(q_\ell) \leq G^K(q_\ell)$, and that both b and b^R are strictly decreasing on $[q_\ell, q_r]$. Then on $[q_\ell, q_r]$, the pointwise first-order optimality conditions obtain, and we have

$$(v(q) - b(q)) F_p^R(b(q)) = F^R(b(q)), \text{ and } \text{-} (v(q) - b(q)) G_b^i(q; b) = 1 - G^i(q; b).$$

Since $b^R(q) = b(q)$ on $[q_\ell, q_r]$, equations 5 imply

$$\frac{F^R(b(q))}{F_p^R(b(q))} = \frac{1 - \mathbb{E}_\xi [K(q + Q(b(q); \xi); b)]}{\mathbb{E}_\xi [K_q(q + Q(b(q); \xi); b) Q_p(b(q); \xi) + K_p(q + Q(b(q); \xi); b)]}. \tag{7}$$

Since b and b^R are strictly decreasing, if there is $q \in [q_\ell, q_r]$ such that $G^R(q) > G^K(q)$, there is $\hat{q} \in [q_\ell, q]$ (since the distributions G^R and G^K are continuous on (q_ℓ, q_r) and $G^R(q_\ell) \leq G^K(q_\ell)$) such that $G^R(\hat{q}) = G^K(\hat{q})$. Applying equations 5 to equation 7 above, $G^K(\hat{q}) = G^R(\hat{q})$ implies $G_q^K(\hat{q}) > G_q^R(\hat{q})$, contradicting $G^R(q) > G^K(q)$. From this it immediately follows that if b^R is strictly decreasing on $[q_\ell, q_r]$ and $G^R(q_\ell) \leq G^K(q_r)$, then $G^R|_{q \in [q_\ell, q_r]} \succeq_{\text{FOSD}} G^K|_{q \in [q_\ell, q_r]}$. Furthermore, since $G^R(0) = G^K(0)$, this also implies that $\lambda(p)$ (the multipliers used to construct F^R at points of discontinuity) are all strictly greater than 1. Let $[q_\ell, q_r]$ be a (maximal) interval on which b is flat. Then $G^R(q_r) \leq G^K(q_r)$. Since this holds at the right endpoint of any flat interval, and given this inequality we have first order stochastic dominance of G^R over G^K for larger quantities, it follows that $G^R \succeq_{\text{FOSD}} G^K$.

Now let b^R be first-order optimal with respect to F^R. By definition, $b^R = b$ wherever b is strictly decreasing and, if b^R is flat on $[q_\ell, q_r]$, it solves the ironed first order condition $(F^R(b^R(q_r)) - \lim_{p \nearrow b(q_r)} F^R(p))(v(q_r) - b(q_r)) = (b^R(q_r) - b(q_r)) F^R(b(q_r))$. Because there is a mass point in F^R at $b^R(q_r)$, this equation cannot be solved at $b^R(q_r) = b(q_r)$, and the flat portion of b^R is longer (extends further to the right) than the constant portion of b. It follows that $b^R \geq b$.

Since b^R under reserve distribution F^R induces a stronger realized quantity distribution, G^R, than b under residual supply distribution G^K, and $b^R \geq b$, it follows that $\pi(b^R; G^R) \geq \pi(b; G^K)$. □

Lemma 5. *Given a reserve distribution F^R with first-order optimal bid b^R and any $\varepsilon > 0$, there is a reserve distribution $\tilde{F}^R$ with a strictly decreasing first-order optimal bid $\tilde{b}^R$ such* *that* $\pi(\tilde{b}^R; \tilde{G}^R) > \pi(b^R, G^R) - \varepsilon$.

Proof. If b^R is strictly decreasing the claim is trivially satisfied. Therefore, assume that b^R is constant on the (maximal) interval $[q_\ell, q_r]$. Let $\tilde{b}^R \leq b^R$ be strictly decreasing and such that $\tilde{b}^R|_{q \notin (q_\ell, q_r]} = b^R|_{q \notin (q_\ell, q_r]}$ and $\tilde{b}^R(q_r) = \lim_{q' \searrow q_r} b^R(q')$. Appealing to the proof of Lemma 4, if $\tilde{F}^R|_{p \geq b^R(q_\ell)} = F^R|_{p \geq b^R(q_\ell)}$, then $\tilde{b}^R$ is a naive best response for all $p \geq b^R(q_\ell)$.

We now show that $\tilde{b}^R$ can be specified so that (i) the probability that $q \in (q_\ell, q_r]$ is lower under $\tilde{b}^R$ than under b^R (thus the probability that $q > q_r$ is higher under $\tilde{b}^R$ than under b^R), (ii) $\tilde{b}^R$ is relatively close to b^R, and (iii) the conditional revenue under $\tilde{b}^R$, given $q \in (q_\ell, q_r]$, is not significantly below the conditional revenue under b^R. First, for a distribution F let $\Delta F \equiv F(\tilde{b}^R(q_\ell)) - F(\tilde{b}^R(q_r))$; since $\tilde{b}^R$ is first-order optimal and is strictly decreasing on $[q_\ell, q_r]$,

$$\begin{aligned}
\Delta \tilde{F}^R &= \left[\exp\left(\int_{\tilde{b}^R(q_r)}^{\tilde{b}^R(q_\ell)} \frac{1}{v\left(\tilde{\varphi}^R(y)\right) - y} dy\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R(q_r)\right) \\
&< \left[\exp\left(\ln\left[v(q_r) - \tilde{b}^R(q_r)\right] - \ln\left[v(q_r) - \tilde{b}^R(q_\ell)\right]\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R(q_r)\right) \\
&= \left(\frac{\tilde{b}^R(q_\ell) - \tilde{b}^R(q_r)}{v(q_r) - \tilde{b}^R(q_\ell)}\right) \tilde{F}^R\left(\tilde{b}^R(q_r)\right) = \left(\frac{\tilde{F}^R\left(\tilde{b}^R(q_r)\right)}{F^R\left(\tilde{b}^R(q_r)\right)}\right) \Delta F^R. \qquad (8)
\end{aligned}$$

The first inequality follows from the fact that v and $\tilde{\varphi}^R$ are strictly decreasing, and the final equality follows from the fact that b^R is first-order optimal with respect to F^R and is flat on $[q_\ell, q_r]$. Now suppose that $\tilde{F}^R(\tilde{b}^R(q_r)) < F^R(\tilde{b}^R(q_r))$; by inequality (8) it must be that $\Delta\tilde{F}^R < \Delta F^R$, and since $\tilde{F}^R(\tilde{b}^R(q_\ell)) = F^R(\tilde{b}^R(q_\ell))$ it follows that $\tilde{F}^R(\tilde{b}^R(q_r)) > F^R(\tilde{b}^R(q_r))$, a contradiction. Then $\tilde{F}^R(\tilde{b}^R(q_r)) \geq F^R(\tilde{b}^R(q_r))$, implying directly that $\Delta\tilde{F}^R \leq \Delta F^R$. Thus point (i) holds for any $\tilde{b}^R$.

Points (ii) and (iii) are shown by construction. For $\delta > 0$ sufficiently small, let $\tilde{b}^R(q_r - \delta) > \tilde{b}^R(q_\ell) - \delta$. Since $\tilde{F}^R|_{p > \tilde{b}^R(q_\ell)} = F^R|_{p > \tilde{b}^R(q_\ell)}$, the expected revenue generated by bid $\tilde{b}^R$ under distribution $\tilde{F}^R$, conditional on $p > \tilde{b}^R(q_\ell)$, is identical to the expected revenue generates by bid b^R under distribution F^R, conditional on $p > \tilde{b}^R(q_\ell)$. Letting $\tilde{b}^R|_{p < \tilde{b}^R(q_r)} = b^R|_{p < \tilde{b}^R(q_r)}$, we have $||\tilde{b}^R - b^R|| < (q_r - q_\ell)\delta + (\tilde{b}^R(q_\ell) - \tilde{b}^R(q_r))\delta$ by construction. By point (i) and the analysis in the proof of Lemma 4, $\tilde{F}^R|_{p < \tilde{b}^R(q_r)} \preceq_{\text{FOSD}} F^R|_{p < \tilde{b}^R(q_r)}$, and so the expected revenue generated by bid $\tilde{b}^R$ under distribution $\tilde{F}^R$, conditional on $p < \tilde{b}^R(q_r)$, is $O(\delta)$ lower than the expected revenue generated by bid b^R under distribution F^R, conditional on $p < \tilde{b}^R(q_r)$. Finally, the utility lost when $p \in [\tilde{b}^R(q_r), \tilde{b}^R(q_\ell)]$ may be bounded in the following way. When $p \in [\tilde{b}^R(q_r), \tilde{b}^R(q_r) - \delta]$ at most quantity δ is lost (versus bid b^R), with marginal utility at most $\overline{v}$; this loss is incurred with at most probability 1, so this loss is bounded above by $\overline{v}\delta$.

When $p \in [\tilde{b}^R(q_\ell) - \delta, \tilde{b}^R(q_\ell)]$, the quantity lost (versus bid b^R) is at most $(q_r - q_\ell) < \overline{Q}$, with marginal utility at most $\overline{v}$. However, the probability that this quantity is lost is bounded by

$$\begin{aligned}
&\tilde{F}^R\left(\tilde{b}^R(q_\ell)\right) - \tilde{F}^R\left(\tilde{b}^R(q_\ell) - \delta\right) \\
&= \left[\exp\left(\int_{\tilde{b}^R(q_\ell)-\delta}^{\tilde{b}^R(q_\ell)} \frac{1}{v\left(\tilde{\varphi}^R(y)\right) - y} dy\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R(q_\ell) - \delta\right) \\
&\le \left[\exp\left(\int_{\tilde{b}^R(q_\ell)-\delta}^{\tilde{b}^R(q_\ell)} \frac{1}{v(q_r) - y} dy\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R(q_\ell)\right) \\
&= \left[\exp\left(\ln\left[v(q_r) - \left(\tilde{b}^R(q_\ell) - \delta\right)\right] - \ln\left[v(q_r) - \tilde{b}^R(q_\ell)\right]\right) - 1\right] \tilde{F}^R\left(\tilde{b}^R(q_\ell)\right) \\
&= \left(\frac{\delta}{v(q_r) - \tilde{b}^R(q_\ell)}\right) \tilde{F}^R\left(\tilde{b}^R(q_\ell)\right).
\end{aligned}$$

Then this probability is also bounded above by a term linear in δ.[72] Then for any $\varepsilon > 0$ there is $\delta > 0$ such that the revenue generated by the first-order optimal bid function $\tilde{b}^R$ under reserve distribution $\tilde{F}^R$ is no more than ε below the revenue generated by the first-order optimal bid function b^R under reserve distribution F^R. □

Corollary 9. *Given any best response bid curve $b(\cdot)$ and any $\varepsilon > 0$, there is a (massless) reserve distribution $\tilde{F}^R$ with strictly decreasing naive best response $\tilde{b}^R$ such that such that the naive best response to F^R generates no more than ε less revenue than $b(\cdot)$.*

Theorem 15. [Uniform Price Revenue Implementation] *Given a massless distribution of reserve prices F^R and a strictly decreasing first-order optimal bid b^R, there is a distribution of reserve prices $\hat{F}^R$ such that the uniform-price auction under reserve distribution $\hat{F}^R$ generates the same expected revenue as the discriminatory auction with first-order optimal bid b^R and reserve distribution F^R.*

Proof. Note first that truthful reporting, $b \equiv v$, is the unique equilibrium in a uniform-price auction with random reserve. Under a random reserve distribution, each bidder's problem is a decision problem, not (strictly speaking) a best response problem. Because demand at a particular price does not affect outcomes at other prices, at each price bidders should demand a utility-maximizing quantity. Thus at each p, $v(\hat{\varphi}^R(p)) = p$.[73]

Revenue in the discriminatory auction under reserve distribution F^R is

$$\mathbb{E}[\pi] = \int_{\underline{b}}^{\overline{b}} \left(p\varphi^R(p) + \int_p^{\overline{b}} \varphi^R(x)\, dx\right) f^R(p)\, dp.$$

[72]Since $b(\cdot) < v(\cdot)$ for all units which are received with strictly positive probability (Lemma 9), $v(q_r) - b^R(q_r) = v(q_r) - \tilde{b}^R(q_\ell) > 0$.

[73]Since b is strictly decreasing and first-order optimal, φ and φ_p are well-defined for all feasible prices p.

Define $\hat{F}^R$ so that

$$\hat{F}^R\left(v\left(\varphi^R\left(p\right)\right)\right) = F^R\left(p;s\right).$$

By construction, $\hat{F}^R_p(v(\varphi^R(p)))v_q(\varphi^R(p))\varphi^R_p(p) = F^R_p(p)$. Additionally, $\operatorname{Supp}\hat{F}^R = [\underline{p},\overline{v}]$, and in a uniform-price auction with reserve distribution $\hat{F}^R$, it is weakly optimal for the bidder to submit truthful bids for all quantities q such that $v(q) \in [\underline{b},\overline{v}]$. The revenue in this auction is

$$\mathbb{E}\left[\hat{\pi}\right] = \int_{\underline{b}}^{\overline{v}} pv^{-1}\left(p\right)\hat{F}^R_p\left(p\right)dp.$$

Apply a change of variables, so that $p = \hat{v}(\varphi^R(p'))$. Then $dp = v_q(\varphi^R(p'))\varphi^R_p(p')dp'$. Since $\varphi^R(\overline{p}) = 0$, this gives

$$\begin{aligned}\mathbb{E}\left[\hat{\pi}\right] &= \int_{\underline{b}}^{\bar{b}} v\left(\varphi^R\left(p'\right)\right)v^{-1}\left(v\left(\varphi^R\left(p'\right)\right)\right)\hat{F}^R_p\left(v\left(\varphi^R\left(p'\right)\right)\right)v_q\left(\varphi^R\left(p'\right)\right)\varphi^R_p\left(p'\right)dp' \\ &= \int_{\underline{b}}^{\bar{b}} v\left(\varphi^R\left(p'\right)\right)\varphi^R\left(p'\right)F^R_p\left(p'\right)dp'.\end{aligned}$$

Then compare,

$$\begin{aligned}\mathbb{E}\left[\pi\right]-\mathbb{E}\left[\hat{\pi}\right] &= \int_{\underline{b}}^{\bar{b}}\left(p\varphi^R\left(p\right)+\int_p^{\bar{b}}\varphi^R\left(x\right)dx\right)F^R_p\left(p\right) - v\left(\varphi^R\left(p\right)\right)\varphi^R\left(p\right)F^R\left(p\right)dp \\ &= \int_{\underline{b}}^{\bar{b}}\left(-\left(v\left(\varphi^R\left(p\right)\right)-p\right)\varphi^R\left(p\right)+\int_p^{\bar{b}}\varphi^R\left(x\right)dx\right)F^R_p\left(p\right)dp \\ &= \int_{\underline{b}}^{\bar{b}}\left(-\left[\frac{F^R\left(p\right)}{F^R_p\left(p\right)}\right]\varphi^R\left(p\right)+\int_p^{\bar{b}}\varphi^R\left(x\right)dx\right)F^R_p\left(p\right)dp \\ &= -\int_{\underline{b}}^{\bar{b}}\varphi^R\left(p\right)F^R\left(p\right)dp + \int_{\underline{b}}^{\bar{b}}\int_p^{\bar{b}}\varphi^R\left(x\right)dxF^R_p\left(p\right)dp \\ &= -\int_{\underline{b}}^{\bar{b}}\varphi^R\left(p\right)F^R\left(p\right)dp + \left[\int_p^{\bar{b}}\varphi^R\left(x\right)dxF^R\left(p\right)\right]\Bigg|_{p=\underline{b}}^{\bar{b}} + \int_{\underline{b}}^{\bar{b}} q^R\left(p\right)F^R\left(p\right)dp \\ &= 0.\end{aligned}$$

The transition from the second line to the third comes from the bidder's first order condition under random reserve. Then the uniform-price auction with reserve distribution $\hat{F}^R$ generates the same revenue as the discriminatory auction with reserve distribution F^R and first-order optimal bid b^R. □

Theorem 16. *Given any supply-reserve distribution K, there is a deterministic quantity*

$Q^\star$ such that the discriminatory auction with fixed supply $Q^\star$ raises greater revenue than the discriminatory auction with supply-reserve distribution K.

Proof. Lemmas 4 and 5, and Theorems 15 and 16, show that (i) any best-response bid in a discriminatory auction with random supply and reserve raises lower revenue than some discriminatory auction with random reserve; (ii) given any $\varepsilon > 0$ and any best-response bid in a discriminatory auction with random reserve, there is another random reserve distribution with a strictly decreasing first-order optimal bid that raises no more than ε less revenue than the original auction; (iii) given any discriminatory auction with a reserve distribution and strictly decreasing first-order optimal bid, there is a uniform-price auction that raises the same revenue; (iv) the uniform-price auction's revenue is maximized by selling the deterministic monopoly quantity. Note that, by points (ii) and (iii), we do not need to consider whether the discriminatory bid under random reserve is fully optimal: it is sufficient to consider only bids which are first-order optimal (and all fully optimal bids are necessarily first-order optimal). Thus to maximize the revenue from a single bidder it is optimal to remove all possible randomness from the system, and sell the bidder the monopoly quantity. Since bidders are symmetric, it follows that it is optimal to (determinstically) sell the aggregate monopoly quantity. □

Corollary 10. *In discriminatory auctions, all mixed strategy equilibria raise weakly lower revenue than the unique equilibrium of the optimal discriminatory auction with deterministic supply.*

Corollary 10 follows immediately from the analysis leading to Theorem 16: the proofs are agnostic as to the source of randomness, thus (in equilibrium) from the perspective of bidder i, randomization on the part of bidder j is indistinguishable from exogenous randomization over elastic supply. Since randomization over elastic supply reduces the seller's revenue below what could be obtained from an optimal deterministic mechanism, it follows that mixed strategy equilibria yield lower expected revenue than the unique equilibrium of the optimal deterministic mechanism.

We have now shown that, absent bidder information, deterministic mechanisms are optimal. Showing dominance of a deterministic mechanism in the presence of bidder information requires a constraint on bidders' demand functions.

Definition 3. [Regular Demand] Let $\mathcal{S} = \{(p^\star, q^\star) \colon \exists s,\ p^\star \in \arg\max_p p v^{-1}(p; s),\ q^\star = v^{-1}(p; s)\}$ be the set of optimal monopoly prices. Bidder values are *regular* if, for any $(p, q), (p', q') \in \mathcal{S}$ if$p' < p$ then $q' < q$.

Values are regular if the monopolist's optimal price and quantity are in monotone correspondence.[74] When values are increasing in signal s, demand is regular when $p+v^{-1}(p;s)/v_p^{-1}(p;s)$ is increasing in s.[75] Thus our regularity condition is similar to the regularity condition in [Myerson, 1981]. When values are regular, the auctioneer can use an elastic supply curve to screen for bidder signal s, and a deterministic elastic supply curve maximizes the seller's revenue.

Proposition 2. [Deterministic Auctions Are Optimal] *When bidder values are regular, revenue in the discriminatory auction is maximized by implementing a deterministic supply curve. Under a deterministic supply curve, every equilibrium in the uniform-price auction raises weakly less revenue than the unique equilibrium in the discriminatory auction.*

Proof. Finally, in a discriminatory auction the optimal quantity is $Q^\star(s) \in \arg\max_Q Q\hat{v}(Q;s)$. In the unique equilibrium of the discriminatory auction, $p^\star(Q^\star(s);s) = \hat{v}(Q^\star(s);s)$. Let $\mathcal{Q} : \mathbb{R}_+ \to \mathbb{R}_+$ be a supply curve, where $\mathcal{Q}(p) = \inf\{Q^\star(s) \colon p^\star(s) > p\}$. Bidder values are regular, so $\mathcal{Q}$ is increasing. Then equilibrium in the discriminatory auction with supply curve $\mathcal{Q}$ is such that for any bidder signal s, $p(Q^\star(s);s) = \hat{v}(Q^\star(s);s)$, and revenue is maximized for each type independently.

The claim of revenue dominance follows from the same arguments used to prove Theorem 9. In the discriminatory auction, equilibrium is unique, and bids equal values at the maximum feasible quantity. In a uniform-price auction, revenue is maximized by selling the monopoly quantity at the monopoly price; since demand is regular, this is a feasible equilibrium. However, equilibrium in the uniform-price auction may be non-unique, and bids may be strictly below values at the maximum feasible quantity.[76] Let $Q(\cdot)$ be a (potentially non-optimal) supply function. Fix a bidder i and consider opponent bid profile $b^{-i} = (b^j)_{j\neq i}$, where for some $\overline{p}$,

$$b^j(q) = \begin{cases} \overline{p} & \text{if } q < q^c, \\ 0 & \text{otherwise.} \end{cases}$$

We consider bidder i submitting a flat bid $b^i = p$, for some $p < \overline{p}$. Then bidder i's realized

[74] Recall that we do not make any assumptions on the bidders' type space, and in particular we do not require that demand increases with type.

[75] To maximize profits, $d[pv^{-1}(p;s)]/dp = 0$, implying $p + v^{-1}(p;s)/v_p^{-1}(p;s) = 0$. If the left-hand side is increasing in s, then $p^\star$ is increasing in s. To have quantity also increasing in s, we need $d[qv(q;s)]dq = 0$, or $qv_q(q;s) + v(q;s) = 0$. Under monopoly, $q = v^{-1}(p;s)$ and $p = v(q;s)$, and the conditions for monotonicity in price and in quantity are equivalent.

[76] The argument below follows the same approach as, e.g., Back+Zender-1993,LiCalzi+Pavan-2005.

quantity is $q(p) = Q(p) - (n-1)q^c$, and her optimization problem is

$$\max_p \int_0^{q(p)} v(x) - p dx \quad \Longrightarrow \quad (v(q(p)) - p)\, q_p(p) - q(p) = 0.$$

To have a symmetric allocation, $q(p) = q^c$, it must be that

$$(v(q^c) - p)\, Q_p(p) = q^c.$$

This equation is not, in general, solved at $v(q^c) = p$ (for example, if the optimal elastic quantity is strictly increasing in price); it follows that bidder i's best response is a constant price below the optimal monopoly price. This allocation can be obtained, without affecting incentives, if all bidders submit bid b,

$$b(q) = \begin{cases} \overline{p} & \text{if } q < q^c, \\ p & \text{otherwise.} \end{cases}$$

It follows that the uniform-price auction with optimal elastic supply generally admits underpricing equilibria. □

Supplementary Appendix: Proofs and Supplementary Results

B Supply and Reserve Decisions under Complete Information

A side product of the proof of Theorem 4 is the following equivalence between reserve prices and a particular change in supply distribution is:

Corollary 11. **[Reserve Price as Supply Restriction]** *Suppose $v(q;s) = v(q)$ for every quantity q and signal s. For every reserve price R there is a reduction of supply that is revenue equivalent to imposing R.*

Without bidder information all reserve prices can be mimicked by supply decisions, but not all supply decisions can be mimicked by the choice of reserve prices. In particular, the revenue with optimal supply is typically higher than the revenue with optimal reserve price. Notice also that, with concentrated distributions, our results imply that attracting an additional bidder is more profitable then setting the reserve price right.

Our analysis of optimum supply in the next subsection (in which we allow informational asymmetry between the seller and the bidders) further implies that:

Corollary 12. **[Optimal Reserve Price]** *Suppose $v(q;s) = v(q)$ for every quantity q and signal s. The optimal reserve price R is equal to bidders' marginal value at the optimal deterministic supply: $R \in \max_{R'} R'v^{-1}(R')$.*

When the reserve price R is binding, the equivalence between reserve prices and supply restrictions gives an implicit "maximum supply" of $\overline{Q}^R = nv^{-1}(R)$. At this quantity, parceled over each agent, each agent's bid will equal her marginal value, as at $\overline{Q}$ in the unrestricted case. Since bids fall below values, this bid is weakly above the bid placed at this quantity when there is no reserve price. For quantities below $\overline{Q}^R$ the c.d.f. is unchanged, hence our representation and uniqueness theorems combine to imply that the bids submitted with a reserve price will be higher than without. These effects can be seen in Figure 4.

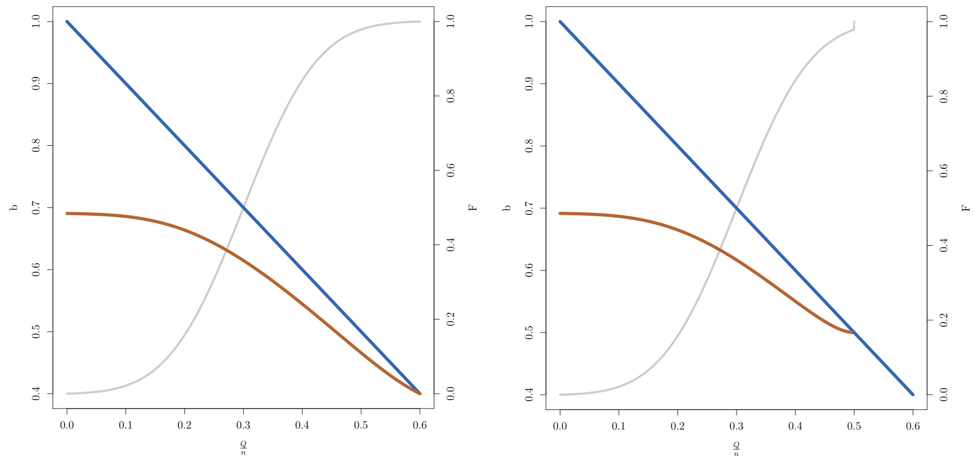

Figure 4: The equilibrium bid function with normal distribution of supply (left), with optimal reserve price (right). The bid for the implicit "maximum quantity" equals the marginal value for this quantity, and the entire bid function shifts up.

C Auxiliary Lemmas

C.1 Endpoint condition (bids match values)

Our equilibrium analysis relies on the identification of the minimum equilibrium market clearing price. In this section we place bounds on this price, and in the case of symmetric bidder information show that this price is uniquely determined. These arguments do not depend on the presence (or absence) of idiosyncratic private information or mixed strategies. We consolidate all bidder-known uncertainty into $\zeta_i = (s, \varepsilon_i, \xi_i)$, where s is the signal observed by all bidders, ε_i is bidder i's idiosyncratic private information, and ξ_i is a term parameterizing bidder i's potentially mixed strategy; thus bidder i's bid $b^i : [0, \overline{Q}] \times \operatorname{Supp} \zeta_i \to \mathbb{R}_+$.[77] Where useful, we consider $\zeta_i | s$ to hold fixed the common signal s while letting ε_i and ξ_i vary.

We also introduce notation for the (essential) minimum market clearing price $\underline{p}$ and (essential) maximum feasible quantity $\overline{q}^i$, conditional on strategy profile $(b^j)_{j=1}^n$,

$$\underline{p}(s) = \operatorname*{ess\,inf}_{Q,\zeta|s} p\left(Q; \left(b^j(\cdot;\zeta_j)\right)_{j=1}^n\right);$$
$$\overline{q}^i(\zeta_i) = \operatorname*{ess\,sup}_{Q,\zeta_{-i}|s} q^i\left(Q; b^i(\cdot;\zeta_i), b^{-i}(\cdot,\zeta_{-i})\right).$$

Thus, when the bidding strategy profile is $(b^j)_{j=1}^n$, the market clearing price is almost never

[77]For compactness we also write $v(\cdot;\zeta_i) = v(\cdot; s, \varepsilon_i)$, but we do not imply that a bidder's marginal value may vary with her action selection from a mixed strategy.

below $\underline{p}(s)$ when the common signal is s, and bidder i's allocation is almost never above $\overline{q}^i(\zeta_i)$ when her type is ζ_i.

Lemma 6. *In any equilibrium, conditional on common signal s, at least $n-1$ bidders, with probability 1, bid their true value for their maximum possible quantity. That is,*

$$\#\left\{i\colon \Pr\left(b^i\left(\overline{q}^i\left(\zeta\right);\zeta\right)=v\left(\overline{q}^i\left(\zeta\right);\zeta\right)\middle|s\right)=1\middle|s\right\}\geq n-1.$$

Proof. For a given agent i, common signal s, and $\lambda>0$, consider an alternative bidding strategy b^λ defined by

$$b^\lambda\left(q;\zeta_i\right)=\begin{cases} b^i\left(q;\zeta_i\right) & \text{if } b^i\left(q;\zeta_i\right)\geq b^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right)+\lambda,\\ \min\left\{b^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right)+\lambda, v\left(q;\zeta_i\right)\right\} & \text{otherwise.}\end{cases}$$

Since $b^i(\cdot;\zeta_i)$ is left-continuous, for small λ this deviation will award the agent all excess quantity above $\sum_{j\neq i}\varphi^j(b^i(\overline{q}^i(\zeta_i);\zeta_i)+\lambda;\zeta_j)$. Let $q^\star(\lambda;\zeta)$ be the quantity obtained under this deviation when, under the original strategy, $q^i(\zeta)$ units would be obtained. Explicitly,

$$q^\star\left(\lambda;\zeta\right)=Q-\sum_{j\neq i}\varphi^i\left(b^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right)+\lambda;\zeta_j\right)=Q-\sum_{j\neq i}\underline{q}^{ji}\left(\lambda;\zeta\right),$$

where $\underline{q}^{ji}(\lambda;\zeta)=\varphi^j(b^i(\overline{q}^i(\zeta_i);\zeta_i)+\lambda;\zeta_j)$ is the quantity bidder j receives when the aggregate signal profile is ζ and bidder i implements bid b^λ; note that $\underline{q}^{ii}(\lambda;\zeta)$ is the maximum quantity for which bidder i bids above $b^i(\overline{q}(\zeta_i);\zeta_i)+\lambda$, which does not depend on ζ_{-i}, and denote this quantity by $\underline{q}^i_\lambda(\zeta_i)$. We will use the quantity $q^\star(\lambda;\zeta)$ to analyze the additional quantity the deviation yields above baseline,

$$\begin{aligned}&\Delta^i_L\left(\lambda;\zeta\right)=q^i\left(\zeta\right)-\underline{q}^{ii}\left(\lambda;\zeta\right),\quad \Delta^i_R\left(\lambda;\zeta\right)==q^\star\left(\lambda;\zeta\right)-q^i\left(\zeta\right),\\ &\Delta^i\left(\lambda;\zeta\right)=\Delta^i_L\left(\lambda;\zeta\right)+\Delta^i_R\left(\lambda;\zeta\right).\end{aligned}$$

Incentive compatibility requires that this deviation cannot be profitable, hence the additional costs must outweigh the additional benefits,

$$\begin{aligned}&\mathbb{E}_{Q,\zeta|s}\left[\int_{\underline{q}^i_\lambda(\zeta_i)}^{q^i(\zeta)} b^\lambda\left(x;\zeta_i\right)-b^i\left(x;\zeta_i\right)dx\middle|q_i\geq \underline{q}^i_\lambda\left(\zeta_i\right)\right]\\ &\quad\geq \mathbb{E}_{Q,\zeta|s}\left[\int_{q^i(\zeta)}^{q^\star(\lambda;\zeta)} v\left(x;\zeta_i\right)-b^\lambda\left(x;\zeta_i\right)dx\middle|q_i\geq \underline{q}^i_\lambda\left(\zeta_i\right)\right].\end{aligned}$$

Importantly, this inequality must hold both ex ante and interim, unconditional on ε_i. Since bids are weaky decreasing, the left-hand expectation is bounded above by

$$\begin{aligned}
&\mathbb{E}_{Q,\zeta|s}\left[\int_{\underline{q}_\lambda^i(\zeta_i)}^{q^i(\zeta)} b^\lambda(x;\zeta_i) - b^i(x;\zeta_i)\,dx\middle| q_i \geq \underline{q}_\lambda^i(\zeta)\right] \\
&\leq \mathbb{E}_{Q,\zeta|s}\left[\int_{\underline{q}_\lambda^i(\zeta_i)}^{q^i(\zeta)} b^i\left(\overline{q}^i(\zeta_i);\zeta_i\right) + \lambda - b^i\left(\overline{q}^i(\zeta_i);\zeta_i\right)\middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right] \\
&= \lambda\mathbb{E}_{Q,\zeta_{-i}|s}\left[\Delta_L^i(\lambda;\zeta)\middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right].
\end{aligned}$$

Since marginal values are Lipschitz in quantity and $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v^i(\overline{q}^i(\zeta_i);\zeta_i)$ by assumption, the right-hand expectation is bounded above by

$$\begin{aligned}
&\mathbb{E}_{Q,\zeta|s}\left[\int_{q^i(\zeta)}^{q^\star(\lambda;\zeta)} v(x;\zeta_i) - b^\lambda(x;\zeta_i)\,dx\middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right] \\
&\geq \mathbb{E}_{Q,\zeta|s}\left[\int_{q^i(\zeta)}^{q^\star(\lambda,\zeta)} \left(v^i\left(\overline{q}^i(\zeta_i);\zeta_i\right) - \left(x - q^i(\zeta)\right)M - \left(b^i\left(\overline{q}^i(\zeta_i);\zeta_i\right)+\lambda\right)\right)_+ dx\middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right] \\
&\geq \mathbb{E}_{Q,\zeta|s}\left[\frac{1}{2}\left(\mu(\zeta_i)-\lambda\right)\min\left\{\Delta_R^i(\lambda;\zeta), \frac{\mu(\zeta_i)-\lambda}{M}\right\}\middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right],
\end{aligned}$$

where $\mu(\zeta_i) = v^i(\overline{q}^i(\zeta_i);\zeta_i) - b^i(\overline{q}^i(\zeta_i);\zeta_i)$. If it is the case that $(\mu(\zeta_i)-\lambda)/M \leq \Delta_R^i(\lambda;\zeta)$ for all λ, then it is impossible that the larger inequality is satisfied for all λ (its left-hand side converges to zero in λ, while the right-hand side converges to a strictly positive value) and incentive compatibility is violated. Therefore we assume that the $\min\{\cdot,\cdot\}$ resolves to $\Delta_R^i(\lambda;\zeta)$. Then the larger inequality implies

$$\lambda\mathbb{E}_{Q,\zeta|s}\left[\Delta_L^i(\lambda;\zeta)\middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right] \geq \mathbb{E}_{Q,\zeta|s}\left[\frac{1}{2}\left(\mu(\zeta_i)-\lambda\right)\Delta_R^i(\lambda;\zeta)\middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right].$$

Since $\Delta_R^i(\lambda;\zeta)$ is bounded, there is $m^i(\lambda)$ such that

$$\lambda\mathbb{E}_{Q,\zeta|s}\left[\Delta_L^i(\lambda;\zeta)\middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right] \geq \frac{1}{2}\left(m^i(\lambda)-\lambda\right)\mathbb{E}_{Q,\zeta|s}\left[\Delta_R^i(\lambda;\zeta)\middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right].$$

For any i, any λ, and any $\kappa > 0$, there is $\Lambda^i(\lambda) > 0$ such that

$$\Lambda^i(\lambda,\kappa) < \frac{1}{2}\left(m^i(\lambda)-\lambda\right)\kappa.$$

The term $m^i(\lambda)$ can be specified so that $m^i(\lambda)-\lambda$ is decreasing in λ, so if $\Lambda^i(\lambda;\kappa) < (m^i(\lambda)-\lambda)\kappa/2$, then $\Lambda^i(\lambda,\kappa) < (m^i(\lambda')-\lambda')\kappa/2$ for all $\lambda' > \lambda$. Then let $\overline{\Lambda} = \min\{\Lambda^i(\lambda,\kappa)\colon\ \Pr_{\zeta_i}(b^i(\overline{q}^i(\zeta_i);\zeta_i) >$

$v^i(\overline{q}^i(\zeta_i);\zeta_i)|s) > 0\}$. For any such $(\kappa, \overline{\Lambda})$, it must be that

$$\kappa \mathbb{E}_{Q,\zeta|s}\left[\Delta_L^i\left(\overline{\Lambda};\zeta\right)\middle|q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right] \geq \mathbb{E}_{Q,\zeta|s}\left[\Delta_R^i\left(\overline{\Lambda};\zeta\right)\middle|q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right].$$

Define bidder j with type ζ_j to be *relevant* given price p (and common signal s) if $b^j(\overline{q}^j(\zeta_j);\zeta_j) \leq p < v^j(\overline{q}^j(\zeta_j);\zeta_j)$. Fixing price p and summing the above incentive inequality over all relevant agents gives

$$\begin{aligned}
&\kappa \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta_L^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] \\
&\geq \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta_R^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] \\
&= \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] - \mathbb{E}_{Q,\zeta|s}\left[\Delta_L^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right].
\end{aligned}$$

Thus,

$$(\kappa+1)\sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta_L^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] \geq \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right].$$

By definition, $\Delta^j(\overline{\Lambda};\zeta) = Q - \underline{q}_\lambda^j(\zeta_j) - \sum_{k \neq j} \underline{q}^{kj}(\overline{\Lambda};\zeta) \equiv Q - \underline{Q}^j(\overline{\Lambda};\zeta)$ and $\Delta_L^j(\overline{\Lambda};\zeta) = q^j(\zeta) - \underline{q}_\lambda^j(\overline{\Lambda};\zeta)$. Furthermore,

$$\sum_{j \text{ relevant}} q^j\left(\zeta\right) - \underline{q}_\lambda^j\left(\zeta_j\right) \leq \sum_j q^j\left(\zeta\right) - \underline{q}_\lambda^j\left(\zeta_j\right) = Q - \underline{Q}\left(\underline{p} + \delta\right).$$

Then it follows that

$$\kappa + 1 \geq \#\left\{j \text{ relevant}\right\}.$$

Since $\kappa > 0$ may be arbitrarily small, it follows that there is at most one relevant bidder; i.e., there is at most a single bidder i such that $\Pr(b^i(\overline{q}^i(\zeta);\zeta) < v(\overline{q}^i(\zeta);\zeta)) < 1$. □

Lemma 7. *For all bidders i and all bidder-common signals s,*

$$\Pr\left(b^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right) = v\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right)\middle|s\right) = 1.$$

Proof. Fix a common signal s. Lemma 6 shows that at least $n-1$ bidders j are such that $b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j)$ with probability 1. If all n bidders' bids satisfy this condition, the desired result follows immediately from market clearing. Otherwise, there is some bidder i such that $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$ with $\zeta_i|s$-strictly positive probability. We show that (i) this bidder's bid must be constant in a neighborhood of $\overline{q}^i(\zeta_i)$, (ii) with $\zeta_{-i}|s$-positive probability, opposing bidders' bids are asymptotically flat near $\overline{q}^j(\zeta_j)$, and (iii) this implies that bidder i has a strict incentive to increase her (flat) bid near $\overline{q}^i(\zeta_i)$.

Let bidder i and parameter ζ_i be such that $b^i(\overline{q}^i(\zeta_i);\zeta_i) = \underline{p} < v(\overline{q}^i(\zeta_i);\zeta_i)$, and assume that b^i is strictly decreasing in a neighborhood to the left of $\overline{q}^i(\zeta_i)$. For $\lambda > 0$, define an alternate bid b^λ,

$$b^\lambda(q) = \begin{cases} b^i(q;\zeta_i) & \text{if } b^i(q;\zeta_i) \geq \underline{p} + \lambda, \\ \underline{p} + \lambda & \text{otherwise.} \end{cases}$$

Since $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$ and we analyze small $\lambda > 0$, we may assume that λ is small enough that for any feasible quantity q, $b^\lambda(q) \leq v(q;\zeta_i)$. Then whenever the market clearing price would be $p < \underline{p} + \lambda$ if bidder i submitted bid b^i, the market clearing price will be $\underline{p} + \lambda$ if she submits bid b^λ instead. Further, bidder i receives the full residual supply,

$$q_i^\lambda = Q - \sum_{j \neq i} \overline{\varphi}^j\left(\underline{p} + \lambda; \zeta_j\right).$$

The utility gain associated with bid b^λ versus bid b^i is bounded below by

$$\mathbb{E}_{Q,\zeta_{-i}}\left[\left.\begin{array}{l}\displaystyle\int_q^{Q-\sum_{j\neq i}\overline{\varphi}^j\left(\underline{p}+\lambda;\zeta_j\right)} v(x;\zeta_i) - \left(\underline{p}+\lambda\right) dx \\ \displaystyle -\int_{\overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)}^{q} \left(\underline{p}+\lambda\right) - b^i(x;\zeta_i)\, dx\end{array}\right| q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right]. \tag{9}$$

Because bidder i's opponents all have $\Pr(b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j) = 1$, and bids are below values and values are Lipschitz continuous, there is $M > 0$ such that $\overline{q}^j(\zeta_j) - \overline{\varphi}^j(\underline{p}+\lambda;\zeta_j) > M\lambda$ with probability 1 for all $j \neq i$. Then, letting $\lambda < v(\overline{q}^i(\zeta_i);\zeta_i) - b^i(\overline{q}^i(\zeta_i);\zeta_i)$, the bound in 9 is in turn bounded below by

$$\begin{aligned}
&\mathbb{E}_{Q,\zeta_{-i}|s}\left[\int_q^{\left(Q-\sum_{j\neq i}\overline{q}^j(\zeta_j)\right)+(n-1)M\lambda} v(x) - \left(\underline{p}+\lambda\right) dx - \left(q - \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right)\lambda \,\middle|\, q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right] \\
&\geq \mathbb{E}_{Q,\zeta_{-i}|s}\left[\left(\left[\left(Q - \sum_{j\neq i}\overline{q}^j(\zeta_j)\right) + (n-1)M\lambda\right] - q\right)\lambda - \left(\left[Q - \overline{Q}\right] + (n-1)M\lambda\right)\lambda \,\middle|\, q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right] \\
&= \mathbb{E}_{Q,\zeta_{-i}|s}\left[\overline{Q} - \sum_{j\neq i}\overline{q}^j(\zeta_j) - q \,\middle|\, q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right]\lambda > 0.
\end{aligned}$$

In the above we rely on the fact that the minimum market clearing price is obtained when aggregate supply is maximized. Since b^λ yields higher expected utility than b^i when $\lambda > 0$

is small, b^i is not a best response, and therefore any best response b^i must be constant in a neighborhood of $\overline{q}^i(\zeta_i)$, if $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$.

Define $\check{q}^i(\zeta_i) = \underline{\varphi}^i(\underline{p};\zeta_i)$ to be the left endpoint of bidder i's bid-flat containing $\overline{q}^i(\zeta_i)$. Without loss of generality, we may assume that $b^i(q;\zeta_i) = \underline{p}$ for all $q > \check{q}^i(\zeta_i)$ whenever $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$: extending the flat portion of the bid function either does not affect allocation, or (by market clearing) increases allocation to some q such that $v(q;\zeta_i) > \underline{p}$. Since $\Pr(b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)|s) > 0$ and $\check{q}^i(\zeta_i) < \overline{q}^i(\zeta_i)$ for all ζ_i with $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$, it follows that $\Pr(p(Q,\zeta) = \underline{p}|s) > 0$. Consider a bidder $j \neq i$ and type ζ_j such that $b^j(\overline{q}^j(\zeta_j);\zeta_j) = \underline{p} = v(\overline{q}^j(\zeta_j);\zeta_j)$; since $\Pr(p(Q,\zeta) = \underline{p}|s) > 0$, it must be that $\Pr(q_j = \overline{q}^j(\zeta_j)|s) > 0$. If the bid $b^j(\cdot;\zeta_j)$ is optimal, it must not be utility-improving to decrease the bid to $b^{\lambda\mu}$, where[78]

$$b^{\lambda\mu}(q) = \begin{cases} b^j(q;\zeta_j) & \text{if } q < \overline{q}^j(\zeta_j) - \lambda, \\ \underline{p} + \mu & \text{otherwise.} \end{cases}$$

The bid $b^{\lambda\mu}$ saves payment $\int_{\overline{q}^j(\zeta_j)-\lambda}^{\overline{q}^j(\zeta_j)} b^j(q;\zeta_j) - (\underline{p}+\mu)dq$ whenever $q_j = \overline{q}^j(\zeta_j)$, but potentially reduces quantity when $q_j \in (\overline{q}^j(\zeta_j) - \lambda, \overline{q}^j(\zeta_j))$. The change in utility from implementing bid $b^{\lambda\mu}$ instead of bid $b^j(\cdot;\zeta_j)$ is bounded below by

$$\begin{aligned} &\int_{\overline{q}^j(\zeta_j)-\lambda}^{\overline{q}^j(\zeta_j)} b^j(q;\zeta_j) - \left(\underline{p}+\mu\right) dq \Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right) \\ -&\int_{\overline{q}^j(\zeta_j)-\lambda}^{\overline{q}^j(\zeta_j)} \int_{\overline{q}^j(\zeta_j)-\lambda}^{q} v(x;\zeta_j) - b^j(x;\zeta_j)\, dx dG^i\left(q;b^j\right). \end{aligned}$$

The derivative of this expression with respect to λ must be weakly negative,

$$\begin{aligned} &\left(b^j\left(\overline{q}^j(\zeta_j) - \lambda;\zeta_j\right) - \left(\underline{p}+\mu\right)\right) \Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right) \\ &- \left(v\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right) - b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right)\right) \Pr\left(q_j \in \left(\overline{q}^j(\zeta_j)-\lambda, \overline{q}^j(\zeta_j)\right)\middle|s\right) \le 0. \end{aligned}$$

This inequality holds for all $\mu > 0$. Letting M be the Lipschitz modulus of v, substituting in for $b^j(\overline{q}^j(\zeta_j);\zeta_j) = \underline{p}$ means that the previous inequality implies

$$\begin{aligned} &\left(b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right) - \underline{p}\right)\Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right) - M\lambda \Pr\left(q_j \in \left(\overline{q}^j(\zeta_j)-\lambda, \overline{q}^j(\zeta_j)\right)\middle|s\right) \le 0 \\ \iff\quad &-\frac{b^j\left(\overline{q}^j(\zeta_j);\zeta_j\right) - b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right)}{\lambda} \le \frac{M\Pr\left(q_j \in \left(\overline{q}^j(\zeta_j)-\lambda, \overline{q}^j(\zeta_j)\right)\middle|s\right)}{\Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right)}. \end{aligned}$$

[78] The μ term ensures that bidder j wins ties against the flat portion of bidder i's bid; this term will be taken to zero.

Taking the limit as $\lambda \searrow 0$, we obtain that $b^j_q(\overline{q}^j(\zeta_j);\zeta_j) = 0$. Thus any bidder $j \neq i$ with type ζ_j such that $b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j)$ and $\Pr(q_j = \overline{q}^j(\zeta_j)|s) > 0$ is such that $b^j_q(\overline{q}^j(\zeta_j);\zeta_j) = 0$.

Now return to bidder i with type ζ_i such that $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$ and $\check{q}^i(\zeta_i) < \overline{q}^i(\zeta_i)$, and consider the alternate bid function b^λ defined in the first portion of this proof. We now place a slightly different bound on the utility gained by implementing bid b^λ versus bid $b^i(\cdot;\zeta_i)$. Payments increase by at most $\overline{Q}\lambda$, with at most probability 1; and, whenever $q_i > \check{q}^i(\zeta_i)$ under $b^i(\cdot;\zeta_i)$, bidder i receives the full residual quantity $Q - \sum_{j\neq i}\overline{\varphi}^j(\underline{p}+\lambda;\zeta_j)$. Then a lower bound on the utility improvement generated by the alternate bid b^λ (versys $b^i(\cdot;\zeta_i)$) is

$$\mathbb{E}_{Q,\zeta_{-i}}\left[\int_q^{Q-\sum_{j\neq i}\overline{\varphi}^j\left(\underline{p}+\lambda;\zeta_j\right)} v\left(x;\zeta_i\right) - \underline{p}dx - \overline{Q}\lambda \middle| q \geq \check{q}^i\left(\zeta_i\right)\right].$$

For b^λ to not be utility-improving, this expectation must be weakly negative. Dividing through by λ and taking the limit at $\lambda \searrow 0$ gives

$$\mathbb{E}_{Q,\zeta_{-i}}\left[-\left(v\left(Q-\sum_{j\neq i}\overline{q}^j\left(\zeta_j\right);\zeta_i\right)-\underline{p}\right)\sum_{j\neq i}\overline{\varphi}^j_p\left(\underline{p};\zeta_j\right)-\overline{Q}\middle| q \geq \check{q}^i\left(\zeta_i\right)\right] \leq 0.$$

By assumption, $v(Q - \sum_{j\neq i}\overline{q}^j(\zeta_j);\zeta_j) > \underline{p}$, and from the previous paragraph we have that $\overline{\varphi}^j_p(\underline{p};\zeta_j) = -\infty$ with strictly positive probability. Then the above inequality cannot be satisfied. It follows that there is no bidder i such that $\Pr(b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)|s) > 0$. □

Corollary 13. *The equilibrium minimum market clearing price is*

$$\underline{p}\left(s\right) = \operatorname*{ess\,inf}_{\zeta|s}\underline{p}\left(\zeta\right), \text{ where } \underline{p}\left(\zeta\right) = \inf\left\{p\colon \sum_{i=1}^n v^{-1}\left(p;\zeta_i\right) \leq \overline{Q}\right\}.$$

Corollary 14. *The equilibrium minimum market clearing price is at least the minimum marginal value for the per-capita maximum quantity,*

$$\operatorname*{ess\,inf}_{\zeta|s} v\left(\frac{1}{n}\overline{Q};\zeta\right) \leq \underline{p}\left(s\right) \leq \operatorname*{ess\,sup}_{\zeta|s} v\left(\frac{1}{n}\overline{Q};\zeta\right).$$

Corollary 15. *When bidders have symmetric information,* $(s,\varepsilon_i) = (s,0)$ *for all bidders* i*, the equilibrium minimum market clearing price equals the marginal value for the per-capita maximum quantity,*

$$\underline{p}\left(s\right) = \hat{v}\left(\overline{Q};s\right).$$

C.2 Pure strategy equilibrium derivation with symmetric bidder information

In this section we present the key lemmas in our results on existence, uniqueness, and bid representation. Under symmetric bidder information, we may assume $\varepsilon_i = 0$ for all bidders i; under pure strategies, we may assume $\xi_i = 0$ for all bidders i.[79] Thus the only exogenous randomness available to bidders when implementing a bid is the common signal s. Because this signal is known to all bidders it is not relevant to their optimization problems, hence we will revert to the shorthand $v(q) = v(q; s) = v(q; \zeta_i)$.

Let us fix a pure-strategy candidate equilibrium $(b^j)_{j=1}^n$. Recall that bid functions are weakly decreasing and (where useful) we may assume that they are right-continuous. Given equilibrium bids the market price (that is, the stop-out price) $p(Q)$ is a function of realized supply Q. In line with Appendix A, denote $G^i(q; b^i) = \Pr(q^i \leq q | b^i)$; that is, $G^i(q; b^i)$ is the probability that agent i receives at most quantity q when submitting bid b^i in the equilibrium considered. The monotonicity of bid functions implies that as long as b^i is an equilibrium bid, and given other equilibrium bids, the probability $G^i(q; b^i)$ depends on b^i only through the value $b^i(q)$.

Our statements in the following results are generally about relevant quantities, such that $G^i(q; b^i) < 1$. For each bidder we ignore quantities larger than the maximum quantity this bidder can obtain in equilibrium; for instance, in the following lemmas, all bidders could submit identical flat bids above their values for units they never obtain.

Lemma 8. *For no relevant price level p are there two or more bidders who, in equilibrium, bid p flat on some non-trivial intervals of quantities.*

Proof. The proof resembles similar proofs in other auction contexts. Suppose agent i bids p on (q^i_ℓ, q^i_r) and bidder j bids p on (q^j_ℓ, q^j_r). Since the support of supply is $\left[0, \overline{Q}\right]$, it must be that $G^i(q^i_r; b^i) > G^i(q^i_\ell; b^i)$ and $G^j(q^j_r; b^j) > G^j(q^j_\ell; b^j)$. Let $\bar{q}^i = \mathbb{E}_Q[q^i | p(Q) = b(q^i_r)]$; without loss of generality, we may assume that agent i is such that $\bar{q}^i < q^i_r$. If $v^i(\bar{q}^i) < b^i(q^i_r)$, the agent has a profitable downward deviation. The agent also has a profitable deviation if $v^i(\bar{q}^i) \geq b^i(q^i_r)$: she can increase her bid slightly by $\lambda > 0$ on $[q^i_\ell, q^i_r)$ (enforcing monotonicity constraints as necessary to the left of q^i_ℓ), keeping her bid below value if necessary.[80] $\square$

Lemma 9. *Bids are below values: $b^i(q) \leq v^i(q)$ for all relevant quantities, and $b^i(q) < v^i(q)$ for $q < \varphi^i(p(\overline{Q}))$.*

[79] The specific values of ε_i and ζ_i are unimportant; all that is essential is that, in the case of pure strategies under symmetric bidder information. there is an isomorphism between $\operatorname{Supp} \zeta_i$ and $\operatorname{Supp} s$.

[80] Because we are conditioning on her expected quantity, we do not need to directly consider whether quantities are relevant.

Proof. Suppose that there exists q with $b^i(q) > v^i(q)$; because b^i is monotonic and v^i is continuous, there must exist a range (q_ℓ, q_r) of relevant quantities such that $b^i(q) > v^i(q)$ for all $q \in (q_\ell, q_r)$. The agent wins quantities from this range with positive probability, and hence the agent could profitably deviate to

$$\hat{b}^i(q) = \min\left\{b^i(q), v^i(q)\right\}.$$

Such a deviation never affects how she might be rationed, by the first part of this proof; hence it is necessarily utility-improving.

Now consider $q < \varphi^i\left(p\left(\overline{Q}\right)\right)$. If $b^i(q) = v^i(q)$ then monotonicity of b^i and Lipschitz-continuity of v^i imply that for small $\varepsilon > 0$ winning units $[q-\varepsilon, q]$ brings per unit profit lower than ε. By lowering the bid for quantities $q' \in [q-\varepsilon, q+\varepsilon]$ to $\hat{b}^i(q') = \min\{v^i(q)-\varepsilon, b^i(q')\}$, the utility loss from losing the relevant quantities is at most $2M\varepsilon^2\left(G_i\left(q+\varepsilon; b^i\right) - G_i\left(q-\varepsilon; b^i\right)\right)$, where M is the Lipschitz modulus of v. Notice that the right-hand probability difference goes to zero as ε goes to zero. At the same time the cost savings from paying lower bids at quantities higher than $q+\varepsilon$ is (at least) of order ε^2. Hence this deviation is profitable, and it cannot be that $b^i(q) = v^i(q)$.

□

Lemma 10. *The market clearing price $p(Q)$ is strictly decreasing in supply Q.*

Proof. We show first that the market clearing price is strictly decreasing in supply for all Q such that $p(Q) > \inf_{Q'} p(Q')$. We then show that p is strictly decreasing at $\overline{Q}$ as long as for any bidder i residual supply $\sum_{j\neq i} \varphi^j(\cdot)$ has nonzero slope at $\underline{p}$. Since Corollary 15 shows that $b^i(\overline{q}^i) = \underline{p}$, Lemma 9 shows that bids are below values, and values are Lipschitz continuous, it follows that residual supply has nonzero slope at $\underline{p}$, and therefore the market clearing price is strictly decreasing in Q.

Since bids are weakly decreasing in quantity, the market price is weakly decreasing as a direct consequence of the market-clearing equation. If price is not weakly decreasing in quantity at some Q, then a small increase in Q will not only increase the price, but will weakly decrease the quantity allocated to each agent. This implies that total demand is no greater than Q, contradicting market clearing.

Define $\underline{p} = \inf_Q p(Q) = p(\overline{Q})$. Lemma 8 is sufficient to imply that the market price must be strictly decreasing for all Q such that $p(Q) > \underline{p}$: at every price level at which at least two bidders pay with positive probability for some quantity, at most one of the submitted bid functions is flat. Furthermore, for no price level $p > \underline{p}$ that with positive probability a bidder pays for some quantity, we can have exactly one bidder, i, submitting a flat bid at price p on an interval of relevant quantities.[81] Indeed, in equilibrium bidder i cannot benefit by slightly reducing the bid on this entire interval; thus it must be that there is some other agent j whose bid function is right-continuous at price p. If $p = 0$, all opponents $j \neq i$ have a profitable deviation.[82] If $p > 0$, we appeal to Lemma 9. Given that i submits a flat bid and the bids of bidder j are strictly below her values for some non-trivial subset of quantities at which her bid is near p, bidder j can then profit by slightly raising her bid; this reasoning is similar to that given in the proof of Lemma 8.

We now show that $p(\cdot)$ is strictly decreasing for all Q. Otherwise, following Lemma 8, there is a bidder i who is submitting a flat bid at $\underline{p}$. Denote the left end of this bidder's flat by $\underline{q}_i = \inf\{q\colon b^i(q) = \underline{p}\}$; by assumption, $\underline{q}_i < \overline{q}_i$.[83] Let $\varepsilon, \lambda > 0$ and define a deviation

$$\hat{b}^{\varepsilon\lambda}(q) = \begin{cases} b^i(q) & \text{if } b^i(q) > \underline{p} + \lambda, \\ \underline{p} + \lambda & \text{if } b^i(q) \leq \underline{p} + \lambda \text{ and } q \leq \underline{q}_i + \varepsilon, \\ \underline{p} & \text{otherwise.} \end{cases}$$

That is, $\hat{b}^{\varepsilon\lambda}$ is b^i, with λ added for length ε at $\underline{q}_i$, and adjusting for the fact that bids must be monotone decreasing. Note that this deviation increases costs by at most $(\varepsilon + (\underline{q}_i - \varphi^i(\underline{p} + \lambda)))\lambda$, with at most probability one. When $q_i \in [\underline{q}_i, \underline{q}_i + \varepsilon]$, it increases the quantity allocation to (approximately) $\max\{\underline{q}_i + \varepsilon, q + \lambda M\}$, where M is the slope of residual supply at the minimum price, $M = \sum_{j \neq i} \varphi^j_p(\underline{p})$.[84] Let $\mu \equiv v^i(\underline{q}_i + \varepsilon) - (\underline{p} + \lambda)$; since bids are below values and values are strictly decreasing, $\mu > 0$ when ε and λ are sufficiently small. Then for the deviation to be nonoptimal, it must be that

$$\begin{aligned} \left(\varepsilon + \left(\underline{q}_i - \varphi^i\left(\underline{p} + \lambda\right)\right)\right)\lambda &\geq \mathbb{E}\left[\left(\max\left\{\varepsilon, q + \frac{\lambda}{M}\right\} - q\right)\mu \middle| q \in \left[\underline{q}_i, \underline{q}_i + \varepsilon\right]\right] \\ &= \mathbb{E}\left[\left(\max\left\{\varepsilon - q, \frac{\lambda}{M}\right\}\right)\mu \middle| q \in \left[\underline{q}_i, \underline{q}_i + \varepsilon\right]\right]. \end{aligned}$$

[81] We refer to any price level p that with positive probability a bidder pays for some quantity, as a relevant price level.

[82] Here we work in a model in which marginal utilities on all possible units is strictly positive. We could dispense with the strict positivity assumption by allowing negative bids.

[83] Because bidders are symmetric, it is not possible that $\overline{q}_i = 0$: in this case, bidder i almost surely receives 0 utility ex post, which is not optimal.

[84] Because we are ultimately letting ε and λ go to zero, this approximation is sufficient. Formally, we may consider $M' < M$ and allow δ to be small enough that the slope of residual supply never falls below M'.

This can be rewritten as

$$\left(\varepsilon+\left(\underline{q}_i-\varphi^i\left(\underline{p}+\lambda\right)\right)\right)\lambda\int_{\underline{q}_i}^{\underline{q}_i+\varepsilon}dF\left(q+\overline{Q}_{-i}\right)\geq\int_{\underline{q}_i}^{\underline{q}_i+\varepsilon}\max\left\{\varepsilon+\underline{q}_i-q,\frac{\lambda}{M}\right\}\mu dF\left(q+\overline{Q}_{-i}\right)$$
$$\geq\int_{\underline{q}_i}^{\underline{q}_i+\varepsilon-\frac{\lambda}{M}}\frac{\mu\lambda}{M}dF\left(q+\overline{Q}_{-i}\right).$$

The $\lambda>0$ multipliers cancel; integrating through gives

$$\left(\varepsilon+\left(\underline{q}_i-\varphi^i\left(\underline{p}+\lambda\right)\right)\right)\left(F\left(\varepsilon+\overline{Q}_{-i}\right)-F\left(\overline{Q}_{-i}\right)\right)\geq\frac{\mu}{M}\left(F\left(\varepsilon-\frac{\lambda}{M}+\overline{Q}_{-i}\right)-F\left(\overline{Q}_{-i}\right)\right).$$

From here the argument is standard. For any $\varepsilon>0$ there is $\lambda>0$ such that $\varepsilon-\lambda/M\geq\varepsilon/2$ and $\underline{q}_i-\varphi^i(\underline{p}+\lambda)<\varepsilon/2$. Thus it must be that

$$\frac{3}{2}\varepsilon\left(F\left(\varepsilon+\overline{Q}_{-i}\right)-F\left(\overline{Q}_{-i}\right)\right)\geq\frac{\mu}{M}\left(F\left(\frac{1}{2}\varepsilon-\overline{Q}_{-i}\right)-F\left(\overline{Q}_{-i}\right)\right)$$
$$\iff\quad F\left(\varepsilon+\overline{Q}_{-i}\right)-F\left(\overline{Q}_{-i}\right)\geq\frac{\mu}{3M}\left[\frac{F\left(\frac{1}{2}\varepsilon-\overline{Q}_{-i}\right)-F\left(\overline{Q}_{-i}\right)}{\frac{1}{2}\varepsilon}\right].$$

This must hold for all $\varepsilon>0$. Taking the limit as $\varepsilon\searrow 0$ gives

$$0\geq\frac{\mu f\left(\overline{Q}_{-i}\right)}{3M}.$$

Since $f(\cdot)>0$, this is a contradiction when$M$ is nonzero. In this case, bidder i has some profitable deviation. □

Corollary 16. *In any pure strategy equilibrium, bid functions are strictly decreasing.*

We define the derivative of G^i with respect to b as follows. For any q and b^i, the mapping $t\mapsto G^i(q;b^i+t)$ is weakly decreasing in t, and hence differentiable almost everywhere. With some abuse of notation, whenever it exists we denote the derivative of this mapping with respect to t by $G^i_b(q;b^i)$.

Lemma 11. *For each agent i and almost every q we have:*

$$G^i_b\left(q;b^i\right)=f\left(q+\sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right)\sum_{j\neq i}\varphi^j_p\left(b^i\left(q\right)\right).$$

Proof. By definition, $G^i(q;b^i) = \Pr(q^i \leq q|b^i)$. From market clearing, this is

$$\begin{aligned} G^i\left(q;b^i\right) =& \Pr\left(Q \leq q + \sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right) \\ =& F\left(q + \sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right). \end{aligned}$$

Where the demands φ^j of agents $j \neq i$ are differentiable, we have

$$G^i_b\left(q;b^i\right) = f\left(q + \sum_{j\neq i}\varphi^j\left(b^i\left(q\right)\right)\right)\sum_{j\neq i}\varphi^j_p\left(b^i\left(q\right)\right).$$

Since for all j, the demand function φ^j must be differentiable almost everywhere, the result follows. □

Lemma 12. *At points where $G^i_b\left(q;b^i\right)$ is well-defined, the first-order conditions of the discriminatory auction are given by*

$$-\left(v\left(q\right) - b^i\left(q\right)\right)G^i_b\left(q;b^i\right) = 1 - G^i\left(q;b^i\right).$$

In the case of pure strategies under symmetric bidder information,[85] *the first-order condition can be written as*

$$-\left(v\left(q\right) - b^i\left(q\right)\right)\left(\frac{d}{db}Q\left(b^i\left(q\right)\right) - \varphi^i_p\left(b^i\left(q\right)\right)\right) = H\left(Q\left(b^i\left(q\right)\right)\right),$$

where $Q\left(p\right)$ is the inverse of $p\left(Q\right)$.

Proof. The agent's maximization problem is given by

$$\max_b \int_0^{\overline{Q}}\int_0^q v\left(x\right) - b\left(x\right)dxdG^i\left(q;b\right).$$

Integrating by parts, we have

$$\max_b - \left[\left(1 - G^i\left(q;b\right)\right)\int_0^q v\left(x\right) - b\left(x\right)dx\right]|^{\overline{Q}}_{q=0} + \int_0^{\overline{Q}}\left(v\left(q\right) - b\left(q\right)\right)\left(1 - G^i\left(q;b\right)\right)dq.$$

[85] The definition of the derivative of bidder i's distribution of supply, G^i_b, obtained in Lemma 11, assumes pure strategies under symmetric bidder information. The first order condition derived here is invariant to the source of randomness in the bidder's allocation, but the statement in terms of aggregate demand holds only for pure strategies under symmetric bidder information.

In the first square bracket term, both multiplicands are bounded for $q \in [0, \overline{Q}]$, hence the fact that $1 - G^i(\overline{Q}; b) = 0$ for all b and $\int_0^0 v(x) - b(x)dx = 0$ for all b allows us to reduce the agent's optimization problem to

$$\max_b \int_0^{\overline{Q}} \left(v\left(q\right) - b\left(q\right)\right)\left(1 - G^i\left(q; b\right)\right) dq.$$

Calculus of variations gives us the necessary condition

$$-\left(1 - G^i\left(q; b^i\right)\right) - \left(v\left(q\right) - b^i\left(q\right)\right) G_b^i\left(q; b^i\right) = 0.$$

This holds at almost all points at which G_b^i is well-defined. Rearrangement yields the first expression for the first-order condition.

To derive the second expression, let us substitute into the above formula for G^i and G_b^i from the Lemma 11. We obtain

$$-\left(v\left(q\right) - b^i\left(q\right)\right) f\left(q + \sum_{j \neq i} \varphi^j\left(b^i\left(q\right)\right)\right)\left(\sum_{j \neq i} \varphi_p^j\left(b^i\left(q\right)\right)\right) = 1 - F\left(q + \sum_{j \neq i} \varphi^j\left(b^i\left(q\right)\right)\right),$$

Now, $Q\left(p\right)$ is well defined since we have shown that p is strictly monotone. By Corollary 16 bids are strictly monotone in quantities and hence $q + \sum_{j \neq i} \varphi^j\left(b^i\left(q\right)\right) = Q\left(b^i\left(q\right)\right)$, and

$$-\left(v\left(q\right) - b^i\left(q\right)\right)\left(\sum_{j \neq i} \varphi_p^j\left(b^i\left(q\right)\right)\right) = H\left(Q\left(b^i\left(q\right)\right)\right).$$

Since $\sum_{j \neq i} \varphi_p^j\left(b^i\left(q\right)\right) = \frac{d}{db} Q\left(b^i\left(q\right)\right) - \varphi_p^i\left(b^i\left(q\right)\right)$, the second expression for the first order condition obtains. □

Lemma 13. *When bidders have symmetric information, equilibrium bidding strategies must be symmetric in all pure strategy equilibria:* $b^i = b$ *for all* i.

Proof. The proof proceeds by establishing an ordering of asymmetric bid functions. We use this ordering to show that equilibrium is symmetric in the $n = 2$ bidder case. The result from the $n = 2$ bidder case provides tools for the general analysis. Intuitively, these results show that agents do not like receiving zero quantity when it is possible to receive a positive quantity; because this is a necessary feature of asymmetric putative equilibria, these bids are not best responses.

Note that for any agent i, $\sum_{j \neq i} \varphi_p^j(p) = Q_p(p) - \varphi_p^i(p)$. Then we can write the agent's first-order condition as

$$b^{i}\left(q\right)=v\left(q\right)+\left(\frac{1-F\left(Q\left(p\right)\right)}{f\left(Q\left(p\right)\right)}\right)\left(\frac{1}{Q_{p}\left(p\right)-\varphi_{p}^{i}\left(p\right)}\right).$$

Now suppose that two agents i, j have bid functions which differ on a set of positive measure; without loss, assume that $b^i > b^j$. Then there is a price p such that $\varphi^i(p) > \varphi^j(p)$, and $v(\varphi^i(p)) < v(\varphi^j(p))$. Substituting into the agents' first-order conditions, this gives

$$\left(\frac{1-F\left(Q\left(p\right)\right)}{f\left(Q\left(p\right)\right)}\right)\left(\frac{1}{Q_{p}\left(p\right)-\varphi_{p}^{i}\left(p\right)}\right)>\left(\frac{1-F\left(Q\left(p\right)\right)}{f\left(Q\left(p\right)\right)}\right)\left(\frac{1}{Q_{p}\left(p\right)-\varphi_{p}^{j}\left(p\right)}\right).$$

Standard rearrangement gives

$$\varphi_{p}^{j}\left(p\right)<\varphi_{p}^{i}\left(p\right).$$

Thus whenever $\varphi^i(p) > \varphi^j(p)$, we have $\varphi^i_p(p) > \varphi^j_p(p)$. Recalling from Corollary 15 that bids must equal values at $q = \overline{Q}/n$, this implies that if there is any p such that $\varphi^i(p) > \varphi^j(p)$, then $\varphi^i > \varphi^j$.

Now consider the implications for the $n = 2$ bidder case, and let $j \neq i$. Assume that there is p with $\varphi^i(p) > \varphi^j(p) > 0$. Then there is some $\check{p}$ such that $\varphi^j(\check{p}) = 0$ and $\varphi^i(\check{p}) > 0$. Basic auction logic dictates that bidder i can never outbid the maximum bid of bidder j (i.e., $b^i(0) = b^j(0)$) thus it must be that bidder i's first-order condition does not apply for initial units, and she is submitting a flat bid. That is, $b^i(q)|_{q\leq\varphi^i(\check{p})} = \check{p}$. Now let $\varepsilon, \lambda > 0$, and define a deviation $\hat{b}^{\varepsilon\lambda}$ for bidder 2,

$$\hat{b}^{\varepsilon\lambda}\left(q\right)=\begin{cases}b^{j}\left(0\right)+\lambda & \text{if } q\leq\varepsilon,\\ b^{j}\left(q\right) & \text{otherwise.}\end{cases}$$

Then for all $q \in (0, \varepsilon]$, $\hat{b}^{\varepsilon\lambda}(q) > b^i(q)$, and when the realized quantity is $Q \in (0, \varepsilon]$ bidder j wins the entire supply. To bound the additional utility, we see that for small $\varepsilon > 0$ bidder j gains at least

$$\int_{0}^{\varepsilon}\left(v\left(x\right)-b^{j}\left(x\right)\right)dx\left(F\left(\varphi^{i}\left(\check{p}\right)\right)-F\left(\varepsilon\right)\right).$$

There is an extra cost paid as well; to bound this cost we will assume that it is paid with probability 1, and this cost is $(b^j(0) + \lambda)\varepsilon - \int_0^\varepsilon b^j(x)dx$. The deviation $\hat{b}^{\varepsilon\lambda}$ is profitable if the ratio of benefits to costs is greater than 1, hence we look at

$$\lim_{\lambda\searrow 0,\varepsilon\searrow 0}\frac{\int_0^\varepsilon\left(v\left(x\right)-b^j\left(x\right)\right)dx\left(F\left(\varphi^i\left(\check{p}\right)\right)-F\left(\varepsilon\right)\right)}{\left(b^j\left(0\right)+\lambda\right)\varepsilon-\int_0^\varepsilon b^j\left(x\right)dx}$$
$$=\lim_{\varepsilon\searrow 0}\frac{\int_0^\varepsilon\left(v\left(x\right)-b^j\left(x\right)\right)dx\left(F\left(\varphi^i\left(\check{p}\right)\right)-F\left(\varepsilon\right)\right)}{b^j\left(0\right)\varepsilon-\int_0^\varepsilon b^j\left(x\right)dx}.$$

The numerator and denominator both go to zero as $\varepsilon\searrow 0$; application of l'Hopital's rule gives

$$=\lim_{\varepsilon\searrow 0}\frac{v\left(0\right)-b^j\left(0\right)}{0}=+\infty.$$

Then either the deviation to $\hat{b}^{\varepsilon\lambda}$ is profitable for bidder j (when $|b^j_q(0)|<\infty$), or bidder i may (essentially) costlessly reduce the initial flat of her bid function (when $|b^j_q(0)|=\infty$).[86]

Now consider the case of $n\geq 3$ agents. By the previous arguments we know that for small quantities submitted bid functions can be ranked (as can their inverses), and that at least two agents submit the highest possible bid function. Thus we focus attention on two selected bid functions,

$$\varphi^H\left(p\right)\equiv\max\left\{\varphi^i\left(p\right)\right\},$$
$$\varphi^L\left(p\right)\equiv\max\left\{\varphi^i\left(p\right):\varphi^i\left(p\right)<\varphi^H\left(p\right)\right\}.$$

Of course, where submitted bid functions are symmetric φ^L will not be well-defined, but because we are attempting to prove that equilibrium bids are symmetric we need only pay attention to the asymmetric case. Lastly, let $m_H\equiv\#\{i\colon\varphi^i=\varphi^H\}$ and $m_L=\#\{i\colon\varphi^i=\varphi^L\}$ be the numbers of agents submitting each bid. As mentioned $m_H\geq 2$, and trivially $m_L\geq 1$; additionally, $m_H+m_L\leq n$. As before, there is $\check{p}$ such that $\varphi^L(\check{p})=0$, $\varphi^H(\check{p})>0$, and $\varphi^L(p)>0$ for all $p<\check{p}$. Corollary 16 shows that φ^H must be continuous, hence the equilibrium first order conditions imply

$$\lim_{p\searrow\check{p}}\left(m_H-1\right)\varphi^H_p\left(p\right)=\lim_{p\nearrow\check{p}}\left(m_H-1\right)\varphi^H_p\left(p\right)+m_L\varphi^L_p\left(p\right).$$

One obvious solution is $\lim_{p\nearrow\check{p}}\varphi^L_p(p)=0$; but since $\varphi^L_p\leq\varphi^H_p\leq 0$ this would imply that bids

[86] Implicit here is that $v(0)>b^j(0)=b^i(0)$, which follows from Lemma 9 but in this particular case is trivial: since bidder i is bidding flat to $\varphi^i(\check{p})$, if $v(0)=b^i(0)$ she is obtaining zero surplus on a positive measure of initial units. She would rather cut her bid and lose all of these units with some probability, saving payment for higher units and *gaining* probable gross utility.

are unboundedly negative, violating monotonicity constraints. Then we have

$$\lim_{p\searrow\check{p}}\varphi_p^H(p)=\lim_{p\nearrow\check{p}}\varphi_p^H(p)+\frac{m_L}{m_H-1}\varphi_p^L(p)<0.$$

Intuitively speaking, the bid function b^H is steeper below $\varphi^H(\check{p})$ than above, and there is a kink at this point. This implies a discontinuity in a bidder L's first-order condition near $q=0$. For p close to but less than $\check{p}$, the first-order condition is

$$\begin{aligned}
&-\left(v\left(\varphi^L(p)\right)-p\right)f\left(Q(p)\right)\left(m_H\varphi_p^H(p)+(m_L-1)\varphi_p^L(p)\right)-\left(1-F\left(Q(p)\right)\right)=0,\\
\implies\quad&-\left(v\left(\varphi^L(p)\right)-p\right)f\left(Q(p)\right)\left((m_H-1)\varphi_p^H(p)+m_L\varphi_p^L(p)\right)-\left(1-F\left(Q(p)\right)\right)>0.
\end{aligned}$$

Letting $p\nearrow\check{p}$, we know that the term $[(m_H-1)\varphi_p^H(p)+m_L\varphi_p^L(p)]$ smoothly[87] approaches $\lim_{p\searrow\check{p}}(m_H-1)\varphi_p^H(p)$, proportional to the marginal probability gained by a slight increase in bid from b^L near $\check{p}$ to $\tilde{b}^L>\check{p}$. Thus, essentially, the L bidder's second-order conditions are not satisfied near $q=0$, and this is not an equilibrium. □

D Proof of Theorem 1 (Uniqueness)

Proof. From Lemma 12 and market clearing, we know that for all bidders

$$\left(p(Q)-v(q)\right)G_b^i\left(q;b^i\right)=1-G^i\left(q;b^i\right).$$

Since Lemma 13 tells us that agents' strategies are symmetric, Lemma 11 allows us to write this as

$$\left(p(Q)-v\left(\frac{1}{n}Q\right)\right)(n-1)\,\varphi_p\left(p(Q)\right)=H(Q).$$

From market clearing, we know that $p(Q)=b(Q/n)$; hence $p_Q(Q)=b_q(Q/n)/n$. Additionally, standard rules of inverse functions give $\varphi_p(p(Q))=1/b_q(Q/n)$ almost everywhere. Thus we have

$$\left(p(Q)-v\left(\frac{1}{n}Q\right)\right)\frac{n-1}{n}=H(Q)\,p_Q(Q).$$

Now suppose that there are two solutions, p and $\hat{p}$. From Corollary 15 we know that $p(\overline{Q})=\hat{p}(\overline{Q})$. Suppose that there is a Q such that $\hat{p}(Q)>p(Q)$; taking Q near the supremum of Q for which this strict inequality obtains we conclude that $\hat{p}_Q(Q)<p_Q(Q)$.[88] But then we

[87] Both φ^H and φ^L are continuous, hence $[(m_H-1)\varphi_p^H+m_L\varphi_p^L]$ and $[m_H\varphi_p^H+(m_L-1)\varphi_p^L]$ are continuous. This additionally implies that φ_p^L and φ_p^H are continuous.

[88] The inequality inversion here from usual derivative-based approaches reflects the fact that we are "working backward" from $\overline{Q}$, while any solution must be weakly decreasing: thus a small *reduction* in Q should

have

$$\hat{p}(Q) > p(Q) = v\left(\frac{1}{n}Q\right) + \left(\frac{n}{n-1}\right) H(Q)\, p_Q(Q) > v\left(\frac{1}{n}Q\right) + \left(\frac{n}{n-1}\right) H(Q)\, \hat{p}_Q(Q).$$

The presumed right-continuity of bids, and hence of p, allows us to conclude that if p solves the first-order conditions, $\hat{p}$ cannot. □

E Proof of Theorem 2 (Bid Representation)

From the first order condition established in the proof of uniqueness, the equilibrium price satisfies

$$p_Q = p\tilde{H} - \hat{v}\tilde{H},$$

where $\hat{v}(x) = v(x/n)$, and $\tilde{H}(x) = [1/H(x)][(n-1)/n]$. The solution to this equation has general form

$$p(Q) = Ce^{\int_0^Q \tilde{H}(x)dx} - e^{\int_0^Q \tilde{H}(x)dx} \int_0^Q e^{-\int_0^x \tilde{H}(y)dy} \tilde{H}(x)\, \hat{v}(x)\, dx$$

parametrized by $C \in \mathbb{R}$. Define $\rho = \frac{n-1}{n} \in [\frac{1}{2}, 1)$. We can see that $\tilde{H} = -\rho \frac{d}{dQ} \ln(1-F)$. Thus we have

$$e^{\int_0^t \tilde{H}(x)dx} = e^{-\rho \int_0^t \partial \ln(1-F(x))dx} = e^{-\rho(\ln(1-F(t)) - \ln 1)} = (1-F(t))^{-\rho}.$$

Substituting and canceling, we have for $Q < \overline{Q}$:

$$p(Q) = \left(C - \rho \int_0^Q f(x)(1-F(x))^{\rho-1} \hat{v}(x)\, dx\right)(1-F(Q))^{-\rho}. \tag{10}$$

Since $1 - F(\overline{Q}) = 0$, this implies that $C = \rho \int_0^Q f(x)(1-F(x))^{\rho-1} \hat{v}(x)\, dx$. The market clearing price is then given by

$$p(Q) = \rho \int_Q^{\overline{Q}} f(x)(1-F(x))^{\rho-1} \hat{v}(x)\, dx \;\; (1-F(Q))^{-\rho}.$$

Since $d/dy[F^{Q,n}(y)] = \rho f(y)(1-F(y))^{\rho-1}(1-F(Q))^{-\rho}$, our formula for market price obtains, and since we have proven earlier that the equilibrium bids are symmetric, the formula for bids obtains as well.

yield $\hat{p}(\overline{Q}) = p(\overline{Q}) \leq p < \hat{p}$.

F Proof and Comments on Theorem 3 (Existence)

While in the main text we present Theorem 3 (existence) as the first result, its proof builds on our Theorems 1 and 2. The proofs of the latter two theorems give conditions on an equilibrium, if one exists, and do not depend on Theorem 3.

Given an inverse bid function φ, define the local inverse hazard rate of residual supply $Y(q;b)$ by

$$Y\left(q;b\right)=\frac{1-F\left(q+\left(n-1\right)\varphi\left(b\right)\right)}{f\left(q+\left(n-1\right)\varphi\left(b\right)\right)}=H\left(q+\left(n-1\right)\varphi\left(b\right)\right).$$

Y is the inverse hazard rate H evaluated at the total quantity demanded at a price of b if one agent demands q units and all others submit the (inverse) bid function φ. The equilibrium existence condition in Theorem 3 can be weakened to the following: there exists a pure-strategy Bayesian-Nash equilibrium whenever, for all $p \in (p(\overline{Q}), p(0))$ and all $Q < \overline{Q} - (n-1)\varphi(p)$,

$$E\left(\frac{1}{n}Q\right)=\left(n-1\right)\left(v\left(\frac{1}{n}Q\right)-p\right)\varphi_{p}\left(p\right)+Y\left(\frac{1}{n}Q;p\right)=0$$
$$\Longrightarrow E_{q}\left(\frac{1}{n}Q\right)=\frac{v_{q}\left(\frac{1}{n}Q\right)Y\left(\varphi\left(p\right);p\right)}{p-v\left(\varphi\left(p\right)\right)}+Y_{q}\left(\frac{1}{n}Q;p\right)>0.$$

The function E represents the equilibrium (negative) first-order conditions in the pay-as-bid auction; E_q is the cross-partial derivative of bidder utility with respect to bid and quantity.[89] Since, by Lemma 9, $v(\varphi(p)) - p > 0$ whenever $p > p(\overline{Q})$ the implication in Theorem 3 is equivalent to

$$Y\left(\varphi\left(p\right);p\right)v_{q}\left(\frac{1}{n}Q\right)-Y_{q}\left(\frac{1}{n}Q;p\right)\left(v\left(\varphi\left(p\right)\right)-p\right)<0.$$

This resembles a standard second-order condition: the marginal gains to increasing the quantity demanded at a particular price are strictly decreasing.

Proof. We want to prove that the candidate equilibrium constructed in Theorem 2 is in fact an equilibrium. Let us this fix a bidder i whose incentives we will analyze, and assume that other bidders follow the strategies of Theorem 2 when bidding on quantities $q \leq \overline{Q}/n$ and that they bid $v(\overline{Q}/n)$ for quantities they never win.[90] Since bids and values are weakly

[89] The cross-partial derivative in this context fills the role of a second derivative in a classical context. If whenever the first-order condition is satisfied — whenever $E(Q/n) = 0$ — the derivative of the first-order condition with respect to its parameter (q) is strictly negative, there can be only one q at which the first-order condition is satisfied for any b. Then there is at most one b at which the first-order condition is satisfied for any q.

[90] When proving the analogue of Theorem 3 in the context of reserve prices, $\overline{Q}$ becomes $nv^{-1}(R)$, the aggregate quantity demanded at the reserve price R. The remainder of the argument does not change.

decreasing, in equilibrium there is no incentive for bidder i to obtain any quantity $q > \overline{Q}/n$ and we only need to check that bidder i finds it optimal to submit bids prescribed by Theorem 2 for quantities $q < \overline{Q}/n$. Thus, agent i maximizes

$$\int_0^{\frac{\overline{Q}}{n}} (v(q) - b(q))(1 - G(q; b))\, dq$$

over weakly decreasing functions $b(\cdot)$.

We need to show that the maximizing function $b(\cdot)$ is given by Theorem 2, and because the bid function in Theorem 2 is strictly monotone, we can ignore the monotonicity constraint.[91] The problem can then be analyzed by pointwise maximization: for each quantity $q \in [0, \overline{Q}/n]$ the agent finds $b(q)$ that maximizes $(v(q) - b(q))(1 - G(q; b))$. Therefore, we can rely on one-dimensional optimization strategies to assert the sufficiency conditions for a maximum. As given in Lemma 12, the agent's first-order condition is

$$-\left(1 - G^i(q; b)\right) - (v(q) - b)\, G_b^i(q; b) = 0.$$

Recall that from any symmetric inverse bid of agent i's opponents, $G_b^i(q; b) = (n-1) f(q + (n-1)\varphi(b))\varphi_p(b)$. Then the first-order condition can be expressed as

$$(n-1)(v(q) - b)\, \varphi_p(b) + Y(q; b) = 0..$$

Suppose that there is $\hat{b}$ that also solves the first-order conditions for the bid for quantity q,[92]

$$(n-1)\left(v(q) - \hat{b}\right)\varphi_p\left(\hat{b}\right) + Y\left(q; \hat{b}\right) = 0.$$

Then since $b(\cdot)$ is continuous and any profitable deviation is such that $\hat{b} \in [b(\overline{Q}/n), b(0)]$ there is some $\hat{q}$ such that $\hat{b} = b(\hat{q})$. At this point,

$$E\left(\hat{q}; \hat{b}\right) \equiv (n-1)\left(v(\hat{q}) - \hat{b}\right)\varphi_p\left(\hat{b}\right) + Y\left(\hat{q}; \hat{b}\right) = 0.$$

If $\partial E/\partial q > 0$ (recall that E is the negative of the first-order condition) whenever $E(q; b) = 0$ then $E(\cdot; b)$ has a unique zero (if it has any). Then there is at most one solution to the first-order conditions; since the bid representation formula in Theorem 2 gives a closed-form

[91] In this regard, Theorem 3 is too strong: the conditions given are sufficient for no bid function — decreasing or otherwise — to generate more utility than the symmetric equilibrium given in Theorem 2.

[92] By the assumption of sufficient demand, bidding $\hat{b} = 0$ is never utility-improving. Further, bidding $\hat{b} > b(0)$ is also not utility-improving, so any solution to the first order conditions can be assumed to be internal.

solution for bids and the first-order conditions have a unique solution, the bids given in the representation theorem are an equilibrium. Calculation gives

$$\frac{\partial E}{\partial q} = (n-1)\, v_q\,(q)\, \varphi_p\,(b) + Y_q\,(q;b) > 0.$$

In the symmetric solution to the market clearing equation we have already seen that $(n-1)\varphi_p(b) = Y(\varphi(b))/(b - v(\varphi(b)))$. Substituting this in gives the desired result. □

G Modifying the Proofs to Allow for Reserve Prices

In Section 3.3 and later we study reserve prices, and we show that imposing a binding reserve price is equivalent to creating an atom at the quantity at which marginal value equals to the reserve price. In order to extend our results to the setting with reserve prices, we thus need to extend them to distributions in which there might be an atom at the upper bound of support $\overline{Q}$. All our results remain true, and the proofs go through without much change except for the end of the proof of Theorem 2, where more care is needed.

The proof of Theorem 2 goes through until the claim that $1 - F(\overline{Q}) = 0$; in the presence of an atom at $\overline{Q}$ this claim is no longer valid. We thus proceed as follows. We multiply both sides of equation (10) by $(1 - F\,(Q))^{\rho}$ and conclude that

$$p\,(Q)\,(1 - F\,(Q))^{\rho} = C - \rho \int_0^Q f\,(x)\,(1 - F\,(x))^{\rho - 1}\, \hat{v}\,(x)\, dx.$$

Now, let $\overrightarrow{F}(\overline{Q}) \equiv \lim_{Q' \nearrow \overline{Q}} F(Q')$. Because the market price and the right-hand integral are continuous as $Q \nearrow \overline{Q}$, we have

$$p\left(\overline{Q}\right)\left(1 - \overrightarrow{F}\left(\overline{Q}\right)\right) = C - \rho \int_0^{\overline{Q}} f\,(x)\,(1 - F\,(x))^{\rho - 1}\, \hat{v}\,(x)\, dx.$$

The parameter C is determined by this equation. The market price function is then

$$p\,(Q) = \left(\frac{1 - \overrightarrow{F}\left(\overline{Q}\right)}{1 - F\,(Q)}\right)^{\rho} p\left(\overline{Q}\right) + \rho \int_Q^{\overline{Q}} f\,(x)\,(1 - F\,(x))^{\rho - 1}\, \hat{v}\,(x)\, dx \;\; (1 - F\,(Q))^{-\rho}. \qquad (11)$$

Recall from Corollary 15 that $p(\overline{Q}) = v(\overline{Q}/n)$. Extending our notation to the auxiliary distribution $F^{Q,n}$, we also have

$$F^{Q,n}(\overline{Q}) - \overrightarrow{F}^{Q,n}(\overline{Q}) = 1 - \overrightarrow{F}^{Q,n}(\overline{Q}) = \left(\frac{1-\overrightarrow{F}\left(\overline{Q}\right)}{1-F\left(Q\right)}\right)^{\rho}.$$

Since $d/dy[F^{Q,n}(y)] = \rho f(y)(1-F(y))^{\rho-1}(1-F(Q))^{-\rho}$ for all $Q, y < \overline{Q}$, we have

$$\begin{aligned} p\left(Q\right) &= \left(F^{Q,n}(\overline{Q}) - \overrightarrow{F}^{Q,n}\left(\overline{Q}\right)\right)\hat{v}\left(\overline{Q}\right) + \int_{Q}^{\overline{Q}} \hat{v}\left(x\right)\frac{d}{dy}\left[F^{Q,n}\left(y\right)\right]_{y=x} dx \\ &= \int_{Q}^{\overline{Q}} \hat{v}\left(x\right) dF^{Q,n}\left(x\right), \end{aligned}$$

proving our formula for equilibrium stop-out price. Theorem 4 follows from noticing that $\overline{Q} = nv^{-1}(R; s)$ will depend on s when bidders have information unavailable to the seller. □

H Proofs for Section 4 Designing Pay-as-Bid Auctions

H.1 Proof of Theorem 5 (Transparency)

Theorem 5 shows that, when the designer is constrained to a reserve price R and a distribution over supply F, the optimal mechanism is deterministic. This is distinct and does not follow from the analysis in Appendix A, which shows that (under regularity conditions on demand) a seller who can implement stochastic elastic supply prefers to implement a deterministic elastic supply curve. In general, fixed supply $Q^{\star}$ and reserve $R^{\star}$ is insufficiently elastic to obtain monopoly rents from all bidder signals s, and a seller who can implement an elastic supply curve will strictly prefer to do so.

Proof of Theorem 5. Consider a pure-strategy equilibrium in a pay-as-bid auction with reserve price R and supply distribution F. In Section 3 we proved that the equilibrium is essentially unique and symmetric. Furthermore, in equilibrium, for any relevant quantity q, each bidder's bid equals the resulting market-clearing price when quantity $Q = nq$ is sold; we denote this market clearing price $p(Q; R, s)$, suppressing in the notation the price's dependence on F as it is constant. We denote the resulting equilibrium revenue by $\pi(Q; R, s)$ and we write $\hat{v}(y; s) = v(y/n; s)$ for a bidder's marginal value from his or her share of quantity sold y.

Proof. The seller maximizes the expected revenue $\mathbb{E}\left[\pi\right] = \mathbb{E}_s \int_0^{\overline{Q}} \pi\left(Q; R, s\right) dF\left(Q\right)$. When bidders' values are low relative to the reserve price, and the realized quantity is high, the reserve price is bidning and the bidders receive only a partial allocation. The equilibrium quantity sold, denoted $Q(y;R,s)$, is thus given by

$$Q\left(y;R,s\right)=\begin{cases}y & \text{if } \hat{v}\left(y;s\right)\geq R,\\ \hat{v}^{-1}\left(R;s\right) & \text{otherwise,}\end{cases}$$

and the expected revenue is

$$\mathbb{E}\left[\pi\right]=\mathbb{E}_{s}\int_{0}^{\overline{Q}}\int_{0}^{Q(y;R,s)}p\left(x;R,s\right)dxdF\left(y\right). \tag{12}$$

Integrating by parts gives

$$\begin{aligned}\mathbb{E}\left[\pi\right]=&\mathbb{E}_{s}\Bigg\{\left[-\left(1-F\left(y\right)\right)\int_{0}^{Q(y;R,s)}p\left(x;R,s\right)dx\right]\Bigg|_{y=0}^{\overline{Q}}\\&+\int_{0}^{\overline{Q}}\left(1-F\left(y\right)\right)p\left(Q\left(y;R,s\right);s\right)dQ\left(y;R,s\right)\Bigg\},\end{aligned}$$

where the first addend is zero. Recognizing that Q is continuous in y and that $Q_y(y;R,s)=1$ for $v(y/n;s)>R$ and $Q_y(y;R,s)=0$ for $v(y/n;s)<R$, writing $Q^*(s)\equiv Q(\overline{Q};R,s)$, and dropping the dependence on fixed R in the notation, we can thus express the expected revenue as

$$\mathbb{E}\left[\pi\right]=\mathbb{E}_{s}\int_{0}^{Q^{*}(s)}\left(1-F\left(y\right)\right)p\left(Q\left(y;s\right);s\right)dy.$$

Our Theorems 2 and 4 give

$$\mathbb{E}\left[\pi\right]=\mathbb{E}_{s}\int_{0}^{Q^{*}(s)}\left(1-F\left(y\right)\right)\left[\left(1-F^{y,n}\left(Q^{*}\left(s\right)\right)\right)\hat{v}\left(Q^{*}\left(s\right);s\right)+\int_{y}^{Q^{*}(s)}\hat{v}\left(x;s\right)dF^{y,n}\left(x\right)\right]dy, \tag{13}$$

where $F^{y,n}\left(x\right)=1-\left(\frac{1-F(x)}{1-F(y)}\right)^{\frac{n-1}{n}}$ is the c.d.f. of the weighting distribution from the theorem.[93] □

[93]The outer integral in equation (13) is bounded to $[0,Q^*(s)]$, thus $y\leq Q^*(s)$ for all y and $F^{y,n}(Q^*(s))$ is well-defined. The left-hand addend in the integral results from the fact that, when $Q^*(s)<\overline{Q}$—that is, when signal-s bidders have low values for the maximum quantity, $\hat{v}(\overline{Q};s)<R$—there is a mass point in the resulting distribution of realized aggregate allocation at $Q^*(s)$; this same expression is seen in equation (11) in Appendix (G).

Applying integration by parts to the inner integral and substituting in for $F^{y,n}$ gives

$$\mathbb{E}\left[\pi\right] = \mathbb{E}_s \int_0^{\overline{Q}} \left(1 - F\left(y\right)\right) \hat{v}\left(y; s\right) + \left(1 - F\left(y\right)\right)^{\frac{1}{n}} \int_y^{Q^*(s)} \hat{v}_q\left(x; s\right) \left(1 - F\left(x\right)\right)^{\frac{n-1}{n}} dxdy. \quad (14)$$

We may change the order of integration of the right-hand double-integral to obtain

$$\begin{aligned}\int_0^{\overline{Q}} \left(1 - F\left(y\right)\right)^{\frac{1}{n}} \int_y^{Q^*(s)} \hat{v}_q\left(x; s\right) \left(1 - F\left(x\right)\right)^{\frac{n-1}{n}} dxdy &= \int_0^{\overline{Q}} \int_0^{x} \left(1 - F\left(y\right)\right)^{\frac{1}{n}} dy \hat{v}_q\left(x; s\right) \left(1 - F\left(x\right)\right)^{\frac{n-1}{n}} dx. \\ &\leq \int_0^{\overline{Q}} x \hat{v}_q\left(x; s\right) \left(1 - F\left(x\right)\right) dx,\end{aligned}$$

where the inequality follows from the facts thta $1 - F(y) \geq 1 - F(x)$ for $y \leq x$ and $\hat{v}_q \leq 0$. Substituting y for x and plugging this bound in the above expression for expected profits, we have

$$\mathbb{E}\left[\pi\right] \leq \mathbb{E}_s \int_0^{\overline{Q}} \left(1 - F\left(y\right)\right) \left(\hat{v}\left(y; s\right) + y \hat{v}_q\left(y; s\right)\right) dy.$$

Notice that $x\hat{v}_q(x; s) + \hat{v}(x; s) = \pi^m_q(x; s)$, where $\pi^m(x; s) = x\hat{v}(x; s)$ is the revenue from selling quantity x at price $\hat{v}(x; s)$. Integrating by parts and denoting

$$Q(x; s) = \min\left\{x, Q^*(s)\right\} = \min\left\{x, Q\left(\overline{Q}; R, s\right)\right\}$$

gives

$$\begin{aligned}\mathbb{E}\left[\pi\right] &\leq \mathbb{E}_s \int_0^{Q^*(s)} \pi^m_q\left(x; s\right) \left(1 - F\left(x\right)\right) dx \\ &= \mathbb{E}_s \pi^m\left(Q^*\left(s\right); s\right) F\left(Q^*\left(s\right)\right) + \int_0^{Q^*(s)} \pi^m\left(x; s\right) dF\left(x\right) \\ &= \mathbb{E}_s \int_0^{\overline{Q}} \pi^m\left(Q\left(x; R, s\right); s\right) dF\left(x\right). \quad (15)\end{aligned}$$

Thus,

$$\mathbb{E}\left[\pi\right] \;\; \leq \;\; \int_0^{\overline{Q}} \mathbb{E}_s\left[\pi^m\left(Q\left(x; R, s\right); s\right)\right] dF\left(x\right).$$

Since there are no cross-terms in this integral, the right-hand side is maximized at a degenerate distribution which maximizes $\mathbb{E}_s[\pi^m(Q(x; R, s); s)]$. But this is exactly the problem of choosing optimal deterministic supply given the reserve price R. It follows that expected revenue is weakly dominated by expected revenue with optimal deterministic supply, hence optimal supply is deterministic. $\square$

Remark 2. The proof of Theorem 5 remains valid for profit maximization of a seller facing increasing marginal costs. Let $C(Q)$ be the seller's cost of supplying quantity Q, and assume that $c(Q) = dC(Q)/dQ$ is positive and weakly increasing. Equation (12) for expected profits in the proof of Theorem 5 must be adjusted to

$$\mathbb{E}[\pi] = \mathbb{E}_s \int_0^{\overline{Q}} \int_0^{Q(y;R,s)} p(x;R,s) - c(x)\, dx dF(y).$$

Subsequent integration by parts remains valid, and equation (14) becomes

$$\mathbb{E}[\pi] = \mathbb{E}_s \int_0^{\overline{Q}} (1 - F(y))(\hat{v}(y;s) - c(y)) + (1 - F(y))^{\frac{1}{n}} \int_y^{Q^*(s)} \hat{v}_q(x;s)(1 - F(x))^{\frac{n-1}{n}} dx dy.$$

As before, letting $\pi^m(q;s,c)$ be monopoly profits when quantity q is sold to type s given marginal cost curve c, we obtain

$$\mathbb{E}[\pi] \leq \mathbb{E}_s \int_0^{\overline{Q}} \pi^m(Q(x;R,s);s,c)\, dF(x).$$

The remainder of the proof is immediate.

H.2 Proof of Theorem 7 (Separable Optimization)

In the proof, we are not restricting attention to signals drawn from a subset of $\mathbb{R}$ and marginal values monotonic in signals. In the general case, define the sets $\underline{\mathcal{S}}(Q,R) = \{s \colon v(Q/n;s) < R\}$ and $\overline{\mathcal{S}}(Q,R) = \{s \colon v(Q/n;s) \geq R\}$ to represent the two possibilities for signal realizations: either the market clearing price is the reserve price, or it exceeds the reserve price. Theorem 7 then takes the form of the claim that the optimal reserve price R^* and quantity Q^* satisfy

$$R^\star \in \arg\max_R R\mathbb{E}\left[v^{-1}(R;s)\middle| s \in \underline{\mathcal{S}}(Q^\star, R^\star)\right]$$

and

$$Q^\star \in \arg\max_Q Q\mathbb{E}\left[v(Q;s)\middle| s \in \overline{\mathcal{S}}(Q^\star, R^\star)\right].$$

Proof. Expected revenue can be expressed as a sum over two integrals,

$$\mathbb{E}_s[\pi] = \int_{s \in \underline{\mathcal{S}}(Q,R)} nR\varphi(R;s)\, d\sigma(s) + \int_{s \in \overline{\mathcal{S}}(Q,R)} Qv\left(\frac{1}{n}Q;s\right) d\sigma(s).$$

From this expression, the seller's choice of optimal (deterministic) quantity and reserve price can be found by taking first-order conditions. Assuming for simplicity that $v(q;\cdot)$ is continuous gives[94]

$$\begin{aligned}\frac{\partial \mathbb{E}_s[\pi]}{\partial R} &= \int_{s\in\underline{\mathcal{S}}(Q,R)} n\varphi(R;s) + nR\varphi_R(R;s)\,d\sigma(s)\\ &\quad + \frac{\partial}{\partial R}\mu\left(\underline{\mathcal{S}}(Q,R)\right)\left[nR\varphi\left(R;\tau(Q,R)\right)\right] + \frac{\partial}{\partial R}\mu\left(\overline{\mathcal{S}}(Q,R)\right)\left[Qv\left(\frac{1}{n}Q;s\right)\right]\\ &= n\int_{s\in\underline{\mathcal{S}}(Q,R)} \frac{\partial}{\partial R}\left[R\varphi(R;s)\right]d\sigma(s).\end{aligned}$$

Similar calculations imply $\partial\mathbb{E}_s[\pi]/\partial Q = \int_{s\in\overline{\mathcal{S}}(Q,R)}(\partial[Qv(Q/n;s)]/\partial Q)d\sigma(s)$. That is, the problem of selecting optimal supply and reserve price is identical to the decoupled problems of maximizing revenue on $s\in\underline{\mathcal{S}}$ by setting a price, and maximizing revenue on $s\in\overline{\mathcal{S}}$ by setting a quantity. □

I Proofs for Section 5 (The Auction Design Game)

In the proofs below we decorate market outcome functions with superscripts denoting the relevant mechanism, where helpful. For example, p^{UPA} is the market-clearing price in the uniform-price auction and p^{PABA} is the market-clearing price in the pay-as-bid auction.

I.1 Proof of Theorem 9

Let us first observe that in a uniform-price auction with deterministic supply, any market clearing price between the reserve price R and the marginal value for the per-capita quantity $v(Q/n;s)$ can obtain in equilibrium.[95]

Lemma 14. [Range of Prices in Uniform-Price Design Game] *Let $p^\star(s)$ denote the market-clearing price in an equilibrium of the uniform-price design game with deterministic supply. Then for all signals s, $p^\star(s)\in[R,\max\{R,v^{-1}(Q/n;s\}]$. Furthermore, for any $p^\star$ such that $p^\star(s)\in[R,\max\{R,v^{-1}(Q/n;s)\}]$ for all s, there is an equilibrium of the uniform-price design game with market-clearing price $p^\star$.*

[94]If $v(q;\cdot)$ were not continuous, the derivatives with respect to the bounds of integration still would cancel: any signal realizations "lost" in the first integral are necessarily "gained" by the second, and vice-versa. Since the definitions of $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ imply that for all $s\in(\text{Cl}\underline{\mathcal{S}}(Q,R))\cap\overline{\mathcal{S}}(Q,R)$, $nR\varphi(R;s)=Qv(Q/n;s)$ the integrand-mass associated with the shifting boundaries is equal in both integrals, hence the terms cancel regardless of the well-behavedness of $v(q;\cdot)$.

[95]In the case of elastic supply, this claim must be constrained. See the discussion at the end of Appendix A.

Proof. To prove the first claim note that $p^\star(s) \geq R$ by definition. If $p^\star(s) > v^{-1}(Q/n;s) > R$, some agent is allocated q_i such that $v(q_i;s) < p^\star(s)$. If she bids $b' = v(\cdot;s)$ instead, she is awarded all units she values above $p^\star(s)$, and possibly more, at a price no greater than $p^\star(s)$. Then she obtains a positive margin on all units received, and possibly decreases her payment; this deviation is profitable.

To prove the second claim, let $p : \operatorname{Supp} s \to [R, \max\{R, v^{-1}(Q/n;s)\}]$ map bidder signals to putative market-clearing prices, and define bids by

$$b\left(q;s,p\right) = \begin{cases} v\left(0;s\right) & \text{if } q < \frac{1}{n}Q, \\ p\left(s\right) & \text{otherwise.} \end{cases}$$

Then downward deviations yield zero quantity, and upward deviations increase the market price without increasing the allocation. Then $(b)_{i=1}^{n}$ is an equilibrium that yields market clearing price $p(s)$. □

Proof of Theorem 9. As discussed in Theorems 5 and 8, we may restrict attention to optimal deterministic supply distributions in both the pay-as-bid and uniform-price auctions. Revenue maximization may then be expressed as a per-agent quantity $q^\star$ and market price $p^\star$; for signals s such that $v(q^\star;s) \geq p^\star$ it is without loss to assume that the total allocation is $nq^\star$—there is sufficient demand for the total quantity at the reserve price—while for signals s such that $v(q^\star;s) < p^\star$ it is clear that some total quantity $nq' < nq^\star$ will be allocated. The seller's expected revenue is then an expectation over bidder signals,

$$\mathbb{E}_s\left[\pi\right] = \mathbb{E}_s\left[nq\left(q^\star, p^\star; s\right) \cdot p\left(q^\star, p^\star; s\right)\right].$$

$q^{\text{UPA}}(q^\star, p^\star; s) = q^{\text{PABA}}(q^\star, p^\star; s)$ — the quantity allocated under the uniform-price auction equals the quantity allocated under the pay-as-bid auction — whenever $v(\cdot;s)$ is strictly decreasing at this quantity, or when $v(\cdot;s) > p^\star$ at this quantity.[96] Since we have assumed that $v(\cdot;s)$ is strictly decreasing, the quantity allocation depends only on $q^\star$ and $p^\star$ and not on the mechanism employed. Additionally, it is the case that $p^{\text{UPA}}(q^\star, p^\star; s) = p^{\text{PABA}}(q^\star, p^\star; s)$ whenever $v(q^\star;s) < p^\star$. Let $\underline{\mathcal{S}}$ be the set of such s,[97]

$$\underline{\mathcal{S}} = \underline{\mathcal{S}}\left(nq^\star, p^\star\right) = \left\{s' : \; v\left(q^\star; s\right) < p^\star\right\}.$$

[96]In the latter case there is excess demand, so all units will be allocated. In the former case all units are allocated at the reserve price; there is a possible difference in allocation when bidders' marginal values are flat over an interval of quantities at the reserve price, since bidders are indifferent between receiving and not receiving these quantities.

[97]This is as defined in Section 4, but with arguments dropped for clarity.

Then we have

$$\mathbb{E}_s[\pi] = p^\star \Pr(s \in \underline{\mathcal{S}})\, \mathbb{E}_s[nq(q^\star, p^\star; s) | s \in \underline{\mathcal{S}}] + nq^\star \Pr(s \notin \underline{\mathcal{S}})\, \mathbb{E}_s[p(q^\star, p^\star; s) | s \notin \underline{\mathcal{S}}].$$

The left-hand term is independent of the mechanism employed, while the right-hand term depends on the mechanism only via the expected market-clearing price. In the pay-as-bid auction, we have seen that $p(q^\star, p^\star; s) = v(q^\star; s)$ for all $s \notin \underline{\mathcal{S}}$, while in the uniform-price auction any price $p \in [p^\star, v(q^\star; s)]$ is supportable in equilibrium. It follows that the pay-as-bid auction weakly revenue dominates the uniform-price auction, and generally will strictly do so. That the seller-optimal equilibrium of the uniform-price auction is revenue-equivalent to the unique equilibrium of the pay-as-bid auction arises from the selection of $p^{\text{UPA}}(q^\star, p^\star; s) = v(q^\star; s)$ for all $s \notin \underline{\mathcal{S}}$. □

To establish comparisons for reserve price and aggregate quantity between auction formats, it is necessary to fully specify the auctioneer's tradeoffs. Since we analyze perfect Bayesian equilibrium, off-path design decisions in the pay-as-bid auction still induce unique bidding strategies (conditional on equilibrium existence). In the uniform-price auction nonuniqueness of equilibrium bidding strategies implies that the auctioneer's tradeoffs from optimizing supply and reserve may also relate to equilibrium selection conditional on supply and reserve. We now define a natural "low-bid" equilibrium that the auctioneer would prefer to avoid.

In the uniform-price auction, equilibrium bidding strategies are unique when the support of supply is sufficiently large.[98] Because any supply distribution F can be nearly costlessly transformed to a distribution with unbounded supply — supply can follow distribution F with probability $1 - \varepsilon$ and an unbounded distribution with probability ε — we model large supply by assuming that supply is unbounded. We define robust uniform-price bids to be those corresponding to an unbounded distribution of supply. Equilibrium in the uniform-price auction is ex post, so these bids are equilibria given any other distribution of supply (holding the reserve price fixed).

Definition 4. [Robust Uniform-Price Bids] Given a reserve price R, *robust uniform-price bids* are

$$b(q; s) = \left(\frac{q}{\hat{Q}(s)}\right)^{n-1} R + (n-1) \int_q^{\hat{Q}(s)} \left(\frac{q}{x}\right)^{n-1} \frac{v(x; s)}{x} dx, \quad \hat{Q}(s) = v^{-1}(R; s).$$

[98]See Klemperer and Meyer [1989]. It is sufficient that the support of supply is $[0, \overline{Q}]$, where $\overline{Q} \geq \sup_s n v^{-1}(R; s)$.

Robust uniform-price bids are a natural limit of slight uncertainty about aggregate supply. We define a bid profile to be robust to uncertainty if small changes in the distribution of quantity result in small changes to equilibrium bids.

Definition 5. [Robust Bids] Given supply distribution F and reserve price R, a bid profile $(b^i)_{i=1}^n$ is *robust to uncertainty* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any supply distribution $\tilde{F}$ with $\sup_{Q \in \mathbb{R}} |F(Q) - \tilde{F}(Q)| < \delta$, all equilibrium bid profiles $(\tilde{b}^i)_{i=1}^n$ are such that $\sup_{s, q \in [0, \hat{Q}(s)]} |b^i(q;s) - \tilde{b}^i(q;s)| < \varepsilon$ for all bidders i, where

$$\hat{Q}(s) = \min \left\{ \max \operatorname{Supp}_F \frac{1}{n} Q, \max \operatorname{Supp}_{\tilde{F}} \frac{1}{n} Q, v^{-1}(R;s) \right\}$$

Proposition 3. [Bids Robust to Uncertainty] *The unique pay-as-bid equilibrium bid profile is robust to uncertainty. Robust uniform-price bids are the unique uniform-price equilibrium bid profile that is robust to uncertainty.*

Proof. Equilibrium pay-as-bid bids are robust to uncertainty because the bid form given in Theorem 2 is continuous in supply distribution. Note that, given any supply distribution F, there is an unbounded supply distribution $\tilde{F}$ with $\sup_{Q \in \mathbb{R}} |F(Q) - \tilde{F}(Q)| < \varepsilon$. With unbounded supply, robust uniform-price bids are the unique bidding equilibrium in the uniform-price auction. It follows that for a bid profile to be robust given supply distribution F and reserve price R, it must be such that $\sup_{s, q \in [0, \hat{Q}(s)]} |b^i(q;s) - \tilde{b}^i(q;s)| < \delta$ for any $\delta > 0$ and all bidders i, where $\tilde{b}^i(\cdot;s)$ is the robust uniform-price bid when bidder type is s. Then $(b^i)_{i=1}^n$ is a robust uniform-price bid profile. □

Robust uniform-price bids are continuous, differentiable, strictly below marginal values for all $q \in (0, \hat{Q}(s))$, and equal to marginal values for $q \in \{0, \hat{Q}(s)\}$. No matter which auction format is employed, optimal supply $Q^\star > 0$. In the pay-as-bid design game the optimal deterministic quantity must be binding for some bidder types, $Q^{\star \text{PAB}} < \sup_s v^{-1}(R;s)$. Since robust uniform-price bids are strictly below value on $(0, Q^{\star \text{PAB}}/n]$ for all types s such that $Q^{\star \text{PAB}} < v^{-1}(R;s)$, the pay-as-bid auction generates strictly greater revenue than the uniform-price auction with robust bidding.[99]

Proposition 4. [Strict Dominance of Pay-as-Bid Revenue] *The pay-as-bid design game generates strictly greater revenue than the unique equilibrium of the uniform-price design game in robust bids.*

[99] Robustness of this form is not possible in the pay-as-bid auction, because bids depend on the distribution of supply (and not just its support). Nonetheless, arguments similar to Corollary 2 imply that small perturbations of the distribution of supply do not affect equilibrium bids very much. In this regard, pay-as-bid bids are robust.

Proposition 4 is key to the proof of Theorem 10 and subsequent results. Robust bids can be used as off-path threats: if the auctioneer implements are particular supply distribution and reserve price, bidders will play the (conditional) seller-optimal equilibrium, and otherwise bidders will play the unique equilibrium in robust uniform-price bids. This implies that any approximately optimal supply distributions and reserve prices can be implemented in a perfect Bayesian equilibrium, as shown in Theorem 10

J Proofs for examples

Linear marginal values with generalized Pareto distribution of supply. For generalized Pareto distributions with parameter $\alpha > 0$,

$$1 - F(x) = \left(1 - \frac{x}{\overline{Q}}\right)^{\alpha}, \qquad f(x) = \frac{\alpha}{\overline{Q}}\left(1 - \frac{x}{\overline{Q}}\right)^{\alpha-1};$$
$$H(x) = \frac{1}{\alpha}\left(\overline{Q} - x\right), \qquad H_q(x) = -\frac{1}{\alpha}.$$

Then with linear market values $v(q) = \beta_0 - q\beta_q$,

$$-\frac{1}{\alpha}\left(\overline{Q} - n\varphi(p)\right)\beta_q + \frac{1}{\alpha}\left(\beta_0 - \varphi(p)\beta_q - p\right) \propto \beta_0 - \left(\overline{Q} - (n-1)\varphi(p)\right)\beta_q - p.$$

For all $Q < \overline{Q}$, $p(Q) > p(\overline{Q})$ and $\overline{Q} > n\varphi(p(Q))$; hence for all $Q < \overline{Q}$,

$$\beta_0 - \left(\overline{Q} - (n-1)\varphi(p)\right)\beta_q - p < \beta_0 - \frac{1}{n}\overline{Q}\beta_q - p\left(\overline{Q}\right) = 0.$$

Then the existence condition is satisfied for all $Q \in [0, \overline{Q})$.

Linear marginal values with generalized Pareto distribution of supply imply linear bids. Recall our bid representation theorem,

$$b(q) = \int_{nq}^{\overline{Q}} v\left(\frac{x}{n}\right) dF^{nq,n}(x).$$

We integrate by parts to find

$$b(q) = \beta_0 - q\beta_q - \frac{\beta_q}{n}\int_{nq}^{\overline{Q}} 1 - F^{nq,n}(x)\,dx.$$

For generalized Pareto distributions with parameter $\alpha > 0$, we have

$$1 - F^{nq,n}(x) = \left(\frac{\overline{Q} - x}{\overline{Q} - nq}\right)^{\alpha\left(\frac{n-1}{n}\right)}.$$

Integrating, the bid function is

$$b(q) = \beta_0 - q\beta_q - \frac{\beta_q}{\alpha(n-1)+n}\left(\overline{Q} - nq\right).$$

Bids are therefore linear in q.

J.1 Optimal supply and reserve with linear demand (Example 1)

The arguments in Section 4 demonstrate that optimal supply and reserve price can be found by separately restricting attention to intervals on which the reserve price or the supply restriction are relevant. For completeness's sake we will not use the separation in this argument, and will work through from the joint maximization problem; using the separation argument would allow us to skip the first several steps.

Assuming that Q and R are both binding, which we will subsequently verify, the monopolist's problem is[100]

$$\max_{Q,R} \int_{\underline{s}}^{\tau} \frac{n}{\rho}(s-R)R\,ds + \int_{\tau}^{\overline{s}} Q\left(s - \frac{\rho Q}{n}\right)ds.$$

Here, $\tau = R + \rho Q/n$. The first-order conditions with respect to Q and R are

$$\begin{aligned}
\frac{\partial}{\partial Q}:\ 0 &= \left[\frac{n}{\rho}(\tau - R)R\right]\frac{\partial \tau}{\partial Q} - \left[Q\left(\tau - \frac{\rho Q}{n}\right)\right]\frac{\partial \tau}{\partial Q} + \int_{\tau}^{\overline{s}} s - \frac{2\rho Q}{n}ds,\\
\frac{\partial}{\partial R}:\ 0 &= \int_{\underline{s}}^{\tau} \frac{n}{\rho}(s-2R)\,ds + \left[\frac{n}{\rho}(\tau - R)R\right]\frac{\partial \tau}{\partial R} - \left[Q\left(\tau - \frac{\rho Q}{n}\right)\right]\frac{\partial \tau}{\partial R}.
\end{aligned}$$

Note that $\tau - R = \rho Q/n$ and $\tau - \rho Q/n = R$; then the $\partial\tau/\partial\cdot$ terms additively cancel, leaving

$$\int_{\tau}^{\overline{s}} s - \frac{2\rho Q}{n}ds = 0, \quad \int_{\underline{s}}^{\tau} \frac{n}{\rho}(s-2R)\,ds = 0.$$

In particular, after cancelation the remaining terms are as given in Theorem 7.

[100]Because the signal distribution is uniform, we ignore the constant of proportionality $1/(\overline{s} - \underline{s})$.

Solving the optimality condition associated with $Q^\star$ gives

$$\frac{1}{2}\left(\overline{s}^2 - \tau^2\right) - \frac{2\rho Q}{n}\left(\overline{s} - \tau\right) = 0.$$

At an internal solution, $\overline{s} > \tau$, so this expression becomes

$$\frac{1}{2}\left(\overline{s} + \tau\right) - \frac{2\rho Q}{n} = 0.$$

Substituting in for $\tau = R + \rho Q/n$ leaves the expression

$$\frac{1}{2}\left(\overline{s} + R + \frac{\rho Q}{n}\right) - \frac{2\rho Q}{n} = 0 \quad \Longrightarrow \quad \frac{3\rho Q}{n} = \overline{s} + R.$$

Solving the optimality condition associated with $R^\star$ gives (removing the constant n/ρ)

$$\frac{1}{2}\left(\tau^2 - \underline{s}^2\right) - 2R\left(\tau - \underline{s}\right) = 0.$$

At an internal solution, $\underline{s} < \tau$, so this expression becomes

$$\frac{1}{2}\left(\tau + \underline{s}\right) - 2R = 0.$$

Substituting in for $\tau = R + \rho Q/n$ leaves the expression

$$\frac{1}{2}\left(R + \frac{\rho Q}{n} - \underline{s}\right) - 2R = 0 \quad \Longrightarrow \quad 3R = \underline{s} + \frac{\rho Q}{n}.$$

Together these equations yield the linear system

$$\begin{aligned} \frac{3\rho Q}{n} =& \overline{s} + R, \\ 3R =& \underline{s} + \frac{\rho Q}{n}. \end{aligned}$$

It is straightforward to see that the solution is

$$Q^\star = \left(\frac{3\overline{s} + \underline{s}}{8\rho}\right) n, \quad R^\star = \frac{\overline{s} + 3\underline{s}}{8}.$$

The signal transition threshold at the optimum is $\tau(Q^\star, R^\star) = (\overline{s} + 3\underline{s})/8 + (3\overline{s} + \underline{s})/8 = (\overline{s}+\underline{s})/2$; then at the optimum both the maximum quantity and the reserve price are binding, as assumed.

The standard monopoly problems are straightforward. The quantity-monopoly problem is

$$\max_{Q} \mathbb{E}_s \left[Qv\left(\frac{Q}{n}; s\right)\right] = \max_{Q} Qv\left(\frac{Q}{n}; \mathbb{E}_s\left[s\right]\right) = \max_{Q} \left(\frac{\overline{s}+\underline{s}}{2} - \rho Q\right) Q.$$

Then optimal quantity is $Q^M = (\overline{s}+\underline{s})/(4\rho)$. The price-monopoly problem is

$$\max_{R} \mathbb{E}_s \left[nR\varphi\left(R; s\right)\right] \propto \max_{R} R\varphi\left(R; \mathbb{E}_s\left[s\right]\right) \propto \max_{Q} \left(\frac{\overline{s}+\underline{s}}{2} - R\right) R.$$

Then optimal price is $R^M = (\overline{s}+\underline{s})/4$.

K Pay-as-Bid vs. Posted Price vs. Cournot Quantity

Recall that in Example 1 we determined optimal reserve price and supply in a Pay-as-Bid auction. Consider now two alternate problems, one in which a standard monopolist posts a price, and one in which the monopolist commits to a quantity. In the former, the monopolist solves

$$\max_{p} n\mathbb{E}_s \left[\frac{1}{\rho}\left(s-p\right)p\right].$$

Then $p^{\text{MONOP}} = (\overline{s}+\underline{s})/4$. In the latter problem, the monopolist solves

$$\max_{q} \mathbb{E}_s \left[\left(s - \frac{\rho q}{n}\right) q\right].$$

Then $q^{\text{MONOP}} = n(\overline{s}+\underline{s})/4\rho$. Comparing the monopolist's problems to the pay-as-bid seller's problem, we can see that $p^{\text{MONOP}} > R^\star$ and $q^{\text{MONOP}} < Q^\star$: that is, the optimally designed pay-as-bid auction allocates a higher quantity at a lower (reserve) price than the classical monopolist's problem.

This comparison turns out to be robust. It arises from the ability of the pay-as-bid seller to hedge the two design parameters against one another. When reserve price is the only instrument available, the seller needs to balance the desire to extract surplus from high-value consumers against the desire to not sacrifice too much quantity with a too-high reserve price against low-value consumers; in the pay-as-bid auction the high-value consumers "self-discriminate," since their unique bid function exactly equals their marginal value when the quantity for sale is deterministic. When quantity is the only instrument available the seller is still balancing the same forces, but the presence of a reserve price ensures that he will not sacrifice too much surplus to low-value consumers when he sets the quantity relatively high. When values are sufficiently regular this argument generalizes in a natural way.[101]

[101] The literature on market regulation has considered whether price or quantity is a better instrument for achieving desired outcomes; the perspective taken is generally that of the regulator, rather than of a

Proposition 5. [Comparison of Pay-as-Bid Seller to Monopolist] *Let quantity-monopoly profits π^Q be given by $\pi^Q(Q,s) = Qv(Q/n;s)$, and let $\hat{Q}(s) \in \arg\max_q \pi^Q(q,s)$; let price-monopoly profits π^R be given by $\pi^R(R;s) = nR\varphi(R;s)$, and let $\hat{R}(s) \in \arg\max_p \pi^R(p;s)$. Let Q^M be optimal quantity-monopoly supply and R^M be optimal price-monopoly reserve against $s \sim \sigma$, and let $Q^\star$ and $R^\star$ be the optimal deterministic supply and reserve price from the pay-as-bid seller's problem. If $v(q;\cdot)$ is monotonically increasing for all q,$\pi^Q(\cdot;s)$ is strictly concave for all s and $\hat{Q}(\cdot)$ is monotonically increasing, then $Q^M \leq Q^\star$; if $v(q;\cdot)$ is monotonically increasing for all q, $\pi^R(\cdot;s)$ is strictly concave for all s and $\hat{R}(\cdot)$ is monotonically increasing, then $R^\star \leq R^M$.*

Proposition 5 is natural in light of the separability of the designer's optimization problem. For a monopolist, increasing a price is typically more desirable when consumers have higher valuations. A supply restriction in effect cuts high-value consumers out of the price optimization problem — provided the price is not too high, their demand will be constrained to available supply — and it is less advantageous to increase prices. Then optimal reserve prices in a supply-constrained pay-as-bid auction will be below optimal monopoly prices. Similar logic, focusing on cutting low-value consumers out of the market, applies in the comparative analysis of market supply.

Proof of Proposition 5. Consider implementing reserve price R; the condition of quantity optimality at $Q^\star(R)$ is

$$0 = \int_{s\in\overline{\mathcal{S}}(Q^\star(R),R)} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma(s).$$

Since $\pi(\cdot;s)$ is strictly concave and $\hat{Q}(\cdot)$ is monotonically increasing, for any Q either $\pi^Q_Q(Q;s) < 0$ for all s, or $\pi^Q_Q(Q;s) > 0$ for all s, or there is some $\bar{s}$ such that $\pi^Q_Q(Q;s') \leq 0$ for all $s' > \bar{s}$ and $\pi^Q_Q(Q;s') \geq 0$ for all $s' > \bar{s}$. Neither of the first two cases support the optimality condition above, hence there is $\bar{s} \in \overline{\mathcal{S}}(Q^\star(R),R)$ such that $\pi^Q_Q(Q^\star(R);s') \leq 0$ for all $s > s'$ and $\pi^Q_Q(Q^\star(R);s') \geq 0$ for all $s < \bar{s}$. Then we have

$$\int_{s\in\overline{\mathcal{S}}(Q^\star(R),R)} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma\left(s\right) \geq \int_{\text{Supp}\sigma} \frac{\partial}{\partial Q}\pi^Q\left(Q^\star\left(R\right);s\right)d\sigma\left(s\right).$$

Since $\pi^Q(\cdot;s)$ is strictly concave for all s, whenever $Q < Q^M$, $\pi^Q_Q(Q;s) > \pi^Q_Q(Q^M;s)$. Then

monopolist. Weitzman [1974] obtains conditions under which price or quantity regulation is preferred under stochastic demand and supply; Roberts and Spence [1976] find that a three-part system involving permits, penalties, and repurchase is preferable to any single-instrument system.

if $Q^{\star} < Q^{M}$, we have

$$\int_{\mathrm{Supp}\sigma} \frac{\partial}{\partial Q} \pi^{Q} \left(Q^{\star} \left(R \right); s \right) d\sigma \left(s \right) > \int_{\mathrm{Supp}\sigma} \frac{\partial}{\partial Q} \pi^{Q} \left(Q^{M}; s \right) d\sigma \left(s \right).$$

Putting these inequalities together gives

$$0 = \int_{s \in \overline{\mathcal{S}}(Q^{\star}(R);R)} \frac{\partial}{\partial Q} \pi^{Q} \left(Q^{\star} \left(R \right); s \right) d\sigma \left(s \right) > \int_{\mathrm{Supp}\sigma} \frac{\partial}{\partial Q} \pi^{Q} \left(Q^{M}; s \right) d\sigma \left(s \right) = 0.$$

This is a contradiction, hence $Q^{\star} \geq Q^{M}$.

A similar argument applies to the case of $R^{\star} \leq R^{M}$. □