Competitive Behavior in Uniform Price Auctions

The Role of Reserve Prices

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Abstract

Uniform price auctions frequently admit equilibria which raise zero seller revenue. We show that when demand is sufficiently strong — when market supply is more than covered by any bidder's opponents — the introduction of a reserve price improves revenue not only by directly increasing the market clearing price, but also by eliminating low revenue equilibria where the market clearing price is almost always equal to the reserve. We provide a full characterization of the existence of low revenue equilibria, in terms of bidder demand at a given reserve price.

1 Introduction

Uniform price auctions for multiple units frequently admit low revenue, collusive-seeming equilibria.[1] In these equilibria bidders implicitly agree on a joint allocation and submit large bids for their individual allocations, independent of their actual values for the items, and zero bids for all other quantities. These bids ensure that the good is allocated deterministically, and the market clearing price is always zero. Reserve prices are implemented to guard against such outcomes: by accepting only bids above some specified threshold, it is certain that whenever the good is sold per-unit revenue will be weakly above the specified price.[2] We show that reserve prices also affect expected revenues through equilibrium selection: the introduction of a reserve price ensures that there is occasionally competition for units, unraveling equilibria where bidders appear to coordinate on the reserve price.

We define a low revenue equilibrium as an equilibrium in which the per-unit price is almost always equal to the reserve price, and show that reserve prices eliminate such equilibria. The mechanism by which these equilibria are eliminated is straightforward. In a low revenue equilibrium without a reserve price, submitted bids are high for all units won and zero for all units lost. When the uniform price is determined by the first rejected bid, the low bids determine the seller's revenue. With the introduction of a reserve price, winning a unit is unprofitable when the bidder's value for the unit is below the reserve price. Then in

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[1] Such equilibria are "collusive-seeming" because they implement natural collusive outcomes through standard Nash equilibria, and do not rely on any collusive agreement.

[2] Binding reserve prices introduce inefficiency in auction outcomes, and an injudiciously selected reserve price can lower expected revenues. We ignore the question of selecting a revenue-maximizing reserve price and focus on the capacity of reserve prices to eliminate low revenue equilibria, similar to equilibrium selection.

equilibrium bids must occasionally be below the reserve price, and bidders are noncompetitors with positive probability. If equilibrium per-unit revenues are almost always equal to the reserve price it must be that equilibrium has the same general structure, where allocations are constant among bidders up to tiebreaking at the reserve. Assuming that any group of $n-1$ bidders has excess demand at the reserve price, and any bidder is noncompetitive with positive probability, with positive probability there is excess demand at all prices above the reserve, and tiebreaking at the reserve price is occasionally necessary. As in other auction settings tiebreaking cannot occur in equilibrium, and low revenue equilibria fail to exist.

Our results are similar to those obtained in models of elastic supply, including LiCalzi and Pavan [2005] and McAdams [2007]. In common-value environments without private information, these papers show that low-revenue equilibria can be eliminated with appropriate supply elasticity. McAdams [2007] further shows that ex post adjustable supply can revenue-dominate any reserve price. Aside from the distinction that we work in a more general signal-value framework, key intuitions pass through to our model.[3] Nonetheless our focus is distinct in that we focus on the elimination of low revenue equilibria through reserve prices alone. This contrasts with results for common value auctions, including Back and Zender [1993], LiCalzi and Pavan [2005], and McAdams [2007], which show that reserve prices *cannot* eliminate certain low revenue equilibria. Back and Zender [1993] shows that low revenue equilibria exist in common-value auctions when the reserve price is below the minimum value for a unit and when there are only $n = 2$ bidders; our results assume the reserve is above the minimum value, and implicitly require that there are $n \geq 3$ bidders. The chief modeling distinction between LiCalzi and Pavan [2005], McAdams [2007] and this paper is their use of a full-information common value context, as opposed to our assumption that bidders retain private information; full information eliminates the possibility that the reserve price causes occasional nonparticipation. In the language of our results, having symmetric information or only two bidders eliminates residual competition.

More broadly our work adds to a literature on modifying multi-unit auctions with small adjustments to eliminate undesirable equilibria. As discussed above, elastic supply can be used to eliminate low-revenue equilibria. These equilibria are also eliminated when the market price is set at the last accepted bid rather than the first rejected bid [Burkett and Woodward, 2018], when the space of acceptable bids is coarse [Kremer and Nyborg, 2004b], when tiebreaking rules are carefully selected [Kremer and Nyborg, 2004a], and when a "buy-it-now" opportunity is available [Tsuchihashi, 2016]. Blume and Heidhues [2004] show that in a single unit second price auction with at least three bidders reserve prices induce truthful reporting, and therefore eliminate low-revenue equilibria; Blume et al. [2009] extend this to the case of the multi-unit Vickrey auction. Since degenerate, single unit first rejected bid uniform price auctions are second price auctions our model clearly ties neatly to Blume and Heidhues [2004].[4]

The existence of zero revenue equilibria in uniform price auctions without reserve prices means that collusive outcomes may not be distinguishable from equilibrium outcomes — under optimal collusion, bidders obtain all units at zero price. For this reason zero revenue equilibria are considered "collusive-seeming" equilibria. Although we establish that uniform price auctions with reserve prices do not admit low revenue equilibrium with the (external) appearance of collusion, we do not make any claim to eliminate formal collusive agreements; see, e.g., the seminal works of Graham and Marshall [1987] and McAfee and McMillan [1992]. Chassang and Ortner [2015] show that constraining bids from above inhibits the ability of participants to tacitly collude. As in our model the modification of the mechanism acts as a tool for equilibrium selection,

[3]The possible elimination of low revenue equilibria via reserve prices is natural in this context, since reserve prices are implementable as infinitely-elastic supply curves.

[4]Blume and Heidhues [2004] show that equilibrium is unique in single-unit second price auctions with a nontrivial reserve price and $n \geq 3$ bidders. We do not obtain any results on equilibrium uniqueness in similar uniform price auctions.

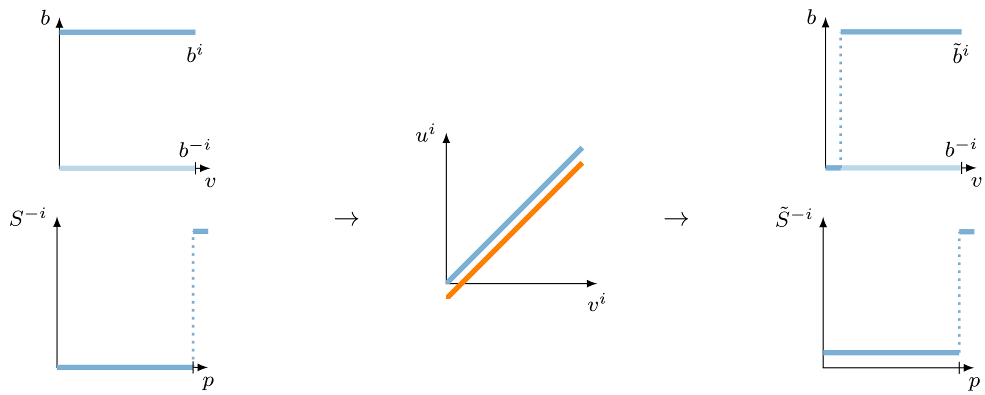

Figure 1: In a standard zero-revenue equilibrium at least one bidder i submits a high bid above all possible values, regardless of her own value profile, and bids are zero for all units not received. Residual supply for i's opponents, S^{-i}, is zero at all prices below the maximum possible valuation. In the presence of a reserve price r, bidder i will pay r for each unit won, rather than zero, and her interim utility curve shifts down and becomes negative in some regions. Her best response in this case is to not participate in the auction when her value is low, implying that her opponents face a nondegenerate residual supply curve. This encourages participation on the part of her opponents.

eliminating "bad" equilibria. A useful implication of our results is that reserve prices can potentially aid in the detection of collusion, even with minimal effect on the operation of the mechanism. Even a small reserve price must, in equilibrium, occasionally yield market prices which are strictly above the reserve, thus a concerned regulator observing a sequence of market prices could test whether bidders are actually colluding.

This paper proceeds in Section 2 by clarifying the breakdown of canonical low-revenue equilibria in the presence of a reserve price. Section 3 sets out our general model, and Section 4 gives our main result: low-revenue equilibria cannot exist in uniform price auctions under a broad range of reserve prices.

2 Example: breakdown of low revenue equilibrium

There are $n \geq 3$ bidders, $i \in \{1, \ldots, n\}$, participating in a first rejected bid uniform price auction for n units. Each bidder i has independent, private value $v_i \sim \mathcal{U}(0,1)$ for each unit she obtains.[5] The auction has reserve price $r \in [0,1]$. Bidder i submits a weakly decreasing bid vector $b_i \geq 0$. Units are allocated to the highest n bids above the reserve price r, and each bidder pays the higher of the $n+1^{\text{th}}$-highest bid and the reserve price for each unit she obtains. Where necessary, ties are broken randomly.[6]

When the reserve price is irrelevant, $r = 0$, there are many zero-revenue equilibria, sharing a common form. Let $b_i = (1, 0, \ldots, 0) \in \mathbb{R}^n_+$ for all bidders i.[7] Then, independent of type realizations, the highest n bids all equal 1 and the $n+1^{\text{th}}$-highest bid is 0. All bidders receive one unit at a price of 0, hence each bidder's interim utility is v_i. This cannot be improved by any deviation: to improve upon receiving a single unit at zero cost would require receiving more than one unit, and to receive more than one unit requires increasing the price per unit to at least 1, which is always above the value for the unit. Then all bidders are

[5]It is straightforward to extend the results in this section to the case of asymmetric distributions and correlation, as long as the support of opponent types is constant in each agent's private signal. We explicitly allow for this in our general results.

[6]As in many auction models, the tiebreaking rule is irrelevant in equilibrium.

[7]In the main results we address the genericity of this form of zero-revenue equilibrium bids.

best-responding. Since the $n+1^{\text{th}}$-highest bid is always zero, the seller's revenue is identically zero.

It is clear that each bidder's strategy must be modified with the introduction of a relevant reserve price $r>0$: bidder i never wants to win an item at a price of r when her value is $v_i<r$. A plausible modification of her original bid is

$$b_i^r(v_i)=\begin{cases}(1,0,\ldots,0) & \text{if } v\geq r,\\ (0,0,\ldots,0) & \text{otherwise.}\end{cases}$$

If bidder i employs bidding strategy b_i^r while her opponents $j\neq i$ retain type-independent strategies b_j, there is excess supply whenever $v_i<r$ — only $n-1$ units are demanded above the reserve price but n units are available. Then any bidder $j\neq i$ can occasionally improve her utility by submitting the bid

$$b_j^r(v)=\begin{cases}(1,r,0,\ldots,0) & \text{if } v\geq r,\\ (0,0,0,\ldots,0) & \text{otherwise.}\end{cases}$$

If all bidders j implement this strategy, ties will need to be broken at price r with positive probability. Tiebreaking implies that some bidder $j\neq 1$ prefers to bid slightly higher and this is sufficient to break equilibrium.

The economic mechanism behind this unraveling is straightforward. Pinning ex post per-unit revenue to the reserve price (or zero, in the case of no sale) requires the use of bids which can dissuade opponents from entering. In the presence of a relevant reserve price this implies a mass point in the stochastic residual demand curve at the reserve price r; in a low-revenue equilibrium it is without loss of generality to assume that there is another mass point at the maximum undominated bid $b=1$. Any bidder with value between two mass points in the allocation function should bid (weakly) between these two points, and tiebreaking implies that no two bidders should share a common mass point in their (individual) bid distribution functions. Then it is not possible for equilibrium to always generate low revenue.

This unraveling need not occur when there are only $n=2$ bidders. In the event that bidder 1's value $v_1<r$ and she prefers to stay out of the auction, bidder 2 will receive every unit for which she submits a bid $b_2\geq r$. Then bidder 2 does not face any meaningful competition at this price, and can submit a bid of r for the second unit and win it whenever bidder 1's value is low. Residual competition is essential to the elimination of low revenue equilibria.

3 Model

A single seller is allocating $Q\in\mathbb{N}$ units in a first rejected bid uniform-price auction with reserve price r.[8] There are $n\geq 3$ bidders, $i\in\{1,\ldots,n\}$, participating in the auction. Bidder i has signal $s_i\sim F^i$ with support $[0,1]^d$, and value profile $v^i(s_i,s_{-i})\in\mathbb{R}_+^Q$ which is coordinatewise weakly decreasing, $v_k^i(s_i,s_{-i})\geq v_{k+1}^i(s_i,s_{-i})$ for all $k\in\{1,\ldots,Q-1\}$.[9] The value profile v^i is monotone increasing in all arguments, and the support of the conditional value profile $v^i(\cdot,s_{-i})$ is a closed subset of $[0,1]^Q$, contains 0, and is constant in s_{-i}; the conditional support of signal $s_i|s_{-i}$ is constant in s_{-i}. Subject to the assumption of conditionally

[8]Our results extend to the case of random and elastic supply. Proving our results with random supply follows essentially the same approach, focusing on the lower bound of the distribution of supply. Proving our results with elastic supply also follows the same approach, but the definition of a low-revenue equilibrium must be altered. Proving our results with elastic and stochastic supply merges these two approaches. We retain the simpler assumption of constant supply for ease of exposition.

[9]It is not essential that v^i represents bidder i's ex post value profile. All results apply equally in the case in which v^i is bidder i's expected value profile prior to the realization of some exogenous randomness.

full support, value profiles may depend arbitrarily on bidder signals, which may in turn have nontrivial correlation.

Value distributions which satisfy these assumptions include:

- Independent private values, $v^i(s) = v^i(s_i)$, generated by independent signals;
- Private values, $v^i(s) = v^i(s_i)$, generated by affiliated signals with conditional full support;
- Semi-interdependent values, where each bidder receives a private value signal s_{i1} and a common value signal s_{i2}, and values have a common value component $\sum_{i=1}^n s_{i2}$ and a private value component s_{i1} that ensures full support;
- Common values, generated by affiliated signals with conditional full support, provided the range of potential values is not restricted by opponent signals.

Because the support of bidder i's value profile $v^i(\cdot, s_{-i})$ contains $0 \in \mathbb{R}_+^Q$ and is independent of s_{-i}, for any bidder i, opponent signal profile s_{-i}, and strictly positive reserve price r, there is strictly positive probability that the bidder's value profile is below the reserve, $\Pr(v^i(s_i, s_{-i}) < r) > 0$. Furthermore, given any set of bidders $\mathcal{I}$, there is strictly positive probabiliy that each bidder in the set simultaneously has a value profile strictly below the reserve.

Bidders submit weakly decreasing bid vectors $b \in \mathbb{R}_+^Q$ to the auctioneer. In the first rejected bid uniform price auction with reserve price $r \geq 0$, market prices are determined by[10]

$$p^\star = \max\left\{r, \inf\left\{p : \#\left\{(i,k) : b_k^i \geq p\right\} \leq Q\right\}\right\}.$$

The market clearing price $p^\star$ is the lowest price at which the market quantity Q is (weakly) underdemanded. This is equal to the $Q+1^{\text{th}}$-highest bid. Bidder i's allocation is determined by her submitted bid and the market-clearing price, and is comprised of her base demand above the market-clearing price plus some additional rationed quantity. Let $\underline{q}_i$ be the base quantity assigned to bidder i and Δ_i be her rationable surplus quantity,

$$\underline{q}_i = \max\left\{k : b_k^i > p^\star\right\}, \ \Delta_i = \#\left\{k : b_k^i = p^\star\right\}.$$

Denote by $G(\cdot; b)$ the distribution over feasible allocations, conditional on submitted bids. For any $q \in$ Supp $G(\cdot; b)$, $\sum_{i=1}^n q_i = \min\{Q, \sum_{i=1}^n q_i + \Delta_i\}$, and for all i, $\underline{q}_i \leq q_i \leq \underline{q}_i + \Delta_i$. Let G^i be the marginal distribution of bidder i's allocation, and assume that G^i depends only on submitted bids. Then bidder i's utility is given by

$$u^i(b; v) = \mathbb{E}_{G^i(\cdot; b)}\left[\sum_{q=1}^{q_i}\left(v_q^i - p^\star\right)\right].$$

Importantly, tiebreaking cannot occur with positive probability in equilibrium: a small deviation will discretely improve at least one bidder's utility.[11]

We analyze Bayesian Nash equilibria: a strategy profile $(b^i)_{i=1}^n$ is a Bayesian Nash equilibrium if $b^i(s_i)$ is s_i-almost surely a best response to $b^{-i} = (b^j)_{j \neq i}$.

[10] In a procurement context this definition must be inverted. Subject to this translation all conclusions go through as expected.

[11] In a standard symmetric tiebreaking model, a small upward deviation will discretely improve any tie-broken bidder's utility, provided bids are below values. With general tiebreaking we can only claim that there exists a bidder with a profitable deviation, but this is sufficient. This claim is formalized in the arguments presented in Section 4.

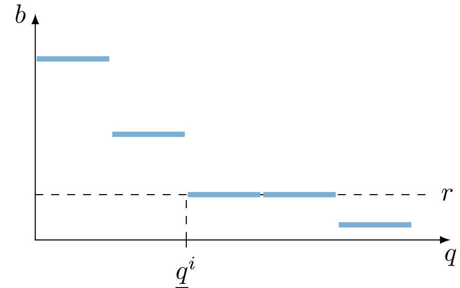

Figure 2: A bid b has (potentially degenerate) regions strictly above and weakly below the reserve price r.

4 Results

Our main results establish that low revenue equilibria cannot exist in the presence of a relevant reserve price. Because sufficiently small reserve prices $r > 0$ trivially rule out zero-revenue equilibria,[12] we define a low revenue equilibrium as one in which the market clearing price, equivalent to the per-unit payment to the seller, is almost always weakly below the reserve price. Thus an equilibrium can provide the seller strictly positive expected revenues while still being low revenue.[13]

Bidder i's *maximum value* for unit k is $\overline{v}_k^i \equiv \sup\{v_k : \Pr(v_k^i(s) \leq v_k) < 1\}$, and her *maximum strict demand* at price p is $\overline{m}^i(p) = \max\{k : \overline{v}_k^i > p\}$.

Definition 1. *A reserve price r* generates excess residual demand *if for all agents i,*

$$\sum_{j \neq i} \overline{m}^j(r) > Q.$$

A reserve price r generates excess residual demand if, given a posted price r, there is positive probability that the aggregate demand of any set of all-but-one bidders exceeds the available supply Q.

Definition 2 (Low revenue)**.** *A strategy profile $(b^i)_{i=1}^n$ is* low revenue *if for s-almost all type profiles $(s_i)_{i=1}^n$,*

$$p^\star\left(b^1(s_1), \ldots, b^n(s_n)\right) = r.$$

Any bid b has (potentially degenerate) ranges on which it is strictly above the reserve price r, on which it equals the reserve price r, and on which it is strictly below the reserve price r. These ranges define a cutoff quantity $\underline{q}^i$ that determines where bids are strictly above the reserve,

$$\underline{q}^i(s_i) = \min\left\{q : \; b^i(q; s_i) > r\right\} \cup \{0\}.$$

Additionally, there is an (endogenous) minimum possible quantity achievable for any bidder. In a monotone equilibrium this will be the quantity obtained when all opponents receive high signals, however even in

[12] The assumption that r is "sufficiently small" is made only to guarantee that bidders sometimes enter the auction. A too-high reserve price trivially generates a zero-revenue equilibrium by causing all bidders to almost surely remain out of the auction.

[13] For example, in a two-bidder second-price auction for a single unit, one bidder can submit a strong bid whenever her value is above the reserve while the other bids the reserve when her value is above the reserve. In this equilibrium, the seller's revenue is always weakly less than the reserve price.

nonmonotone equilibria [McAdams, 2007] such a quantity, $q^i_{\min}$, will still exist,

$$q^i_{\min}(s_i) = \min\left\{q : \Pr_{s_{-i}}\left(q^i(s_i, s_{-i}) \le q\right) > 0\right\}.$$

In a low-revenue equilibrium there is a natural relationship between $q^i_{\min}$ and $\underline{q}^i$: since the market clearing price is almost always below r and is set by the highest non-accepted bid, it must be that $\underline{q}^i \le q^i_{\min}$.

Lemma 1 (Low-revenue allocations)**.** *In a low-revenue equilibrium,* $\underline{q}^i(s_i) \le q^i_{min}(s_i)$ *for almost all* s_i*.*

Proof. Suppose otherwise. Then $\Pr_{s_{-i}}(q^i(s_i, s_{-i}) < \underline{q}^i(s_i)) > 0$ for a positive-probability set of s_i. For any such (s_i, s_{-i}), it must be that $p^\star \ge b^i_{\underline{q}^i(s_i)}(s_i) > r$, implying that $p^\star > r$ with strictly positive probability, contradicting low-revenue equilibrium. □

Corollary 1 (Low-revenue cutoffs)**.** *In any low-revenue equilibrium,* $\Pr\left(\sum_{i=1}^n \underline{q}^i(s_i) \le Q\right) = 1$.

Since $\sum_{i=1}^n q^i(s) \le Q$ and $q^i_{\min}(s_i) \le q^i(s)$ for all s, Corollary 1 follows immediately from Lemma 1.

The ability of reserve prices to eliminate low-revenue equilibria hinges on the fact that bidders want to submit low bids when they know their value is almost certainly below the reserve.[14] To support a canonical *zero*-revenue equilibrium, some bidders must commit to bids which are higher than almost all values. Without a reserve price, winning the auction at an unrealistically high bid is utility-positive regardless of the bidder's own type. In the presence of a reserve price, when the bidder's value is extremely low she will be better off remaining out of the auction altogether. Thus in a low-revenue equilibrium each bidder is a nonparticipant with strictly positive probability.

Lemma 2 (Non-participation)**.** *In any equilibrium with a binding reserve* $r > 0$*, for all bidders* i*,* $b^i_k(s_i) < r$ *for all units* k *with positive probability.*

Proof. Define s_i to be *low value* if $\Pr_{s_{-i}}(v^i(s_i, s_{-i}) < r) = 1$. By assumption of conditional full support of values, s_i is low value with strictly positive probability. Since market prices are bounded below by the reserve price, a low value bidder strictly prefers bidding below the reserve to winning any positive allocation. Then if a low value bidder participates in the auction and bids weakly above the reserve, she must win with zero probability. Since each bidder is low value with strictly positive probability, it follows that low value bidders are almost surely bidding above the reserve price. Then some low value bidder is winning a strictly positive quantity with strictly positive probability, and can improve her utility by remaining out of the auction. It follows that all low value bidders remain out of the auction, establishing the desired result. □

Because values have conditional full support, if r generates excess residual demand there is positive probability that all bidders have values strictly above the reserve price r. And because signals and values have conditional full support some bidder who receives a strictly positive allocation when her value is high will, with positive probability, have a low value even when her opponents' values remain high. Then the quantity she would have received if her value were high must be reallocated among her competitors, who compete to win the quantity. This implies that there is no low revenue equilibrium.

Theorem 1 (Nonexistence of low-revenue equilibrium)**.** *If the reserve price* $r > 0$ *generates excess residual demand there is no low-revenue equilibrium in the first rejected bid uniform price auction.*

[14] A similar argument can be made with respect to equilibrium beliefs about values conditional on winning. Arguments are simplified by restricting attention to agents with almost-surely low values.

Proof. Define s_i to be *high value* if $\Pr_{s_{-i}}(v_k^i(s_i, s_{-i}) > r, \ \forall k \leq \overline{m}^j(r)) = 1$. Because the reserve price r generates excess residual demand, the assumption of conditional full support ensures that each agent i is high value with strictly positive probability, the set of high value signal profiles $\overline{S}$ has strictly positive probability. Let $q(\overline{S})$ be the set of equilibrium allocations achievable when signals are in $\overline{S}$. Because signals are in $\overline{S}$ with positive probability and allocations are discrete, there is some allocation $q^\star \in q(\overline{S})$ that occurs with strictly positive probability, both conditional on $s \in \overline{S}$ and unconditionally.

Let P be the set of agents receiving a strictly positive allocation at $q^\star$, $P = \{i : \ q_i^\star > 0\}$. Market supply is positive, $Q > 0$, so $P \neq \emptyset$. Let $k \in \arg\max_{k' \in P} q_{k'}^\star$, and consider the set of signal profiles $\underline{S}$,

$$\underline{S} = \left\{ (\tilde{s}_k, s_{-k}) : \ s \in \overline{S}, \ q\left(s\right) = q^\star, \ \text{and } b^k\left(\tilde{s}_k\right) < r \right\}.$$

That is, $\underline{S}$ is the set of signal profiles that, but for the fact that bidder k is low value, are high value and would generate the allocation $q^\star$. Lemma 2 implies that $\Pr(s \in \underline{S}) > 0$. Because the reserve price r generates excess residual supply, there is an agent i such that $q^i(s) < \overline{m}^i(r)$ with positive probability for $s \in \underline{S}$; let U be the set of such agents. Let U' be the set of agents j such that $q_j^\star < q^j(s)$ with positive probability on $\underline{S}$; by Corollary 1, $b^j_{q_j^\star+1}(s_j) \leq r$ for all $j \in U'$. Excess residual demand and the fact that k maximizes $q_{k'}^\star$ imply that it is possible to select $i \in U$ and $j \in U'$ with $i \neq j$. Then $b^j_{q^j(s)}(s_j) = r = b^i_{q^i(s)+1}(s_i) < v^i(s_i, s_{-i})$ with positive probability on $\underline{S}$. Then for any $\varepsilon > 0$ bidder i can increase her bid to $b^i(s_i) + \varepsilon$ and ensure that she wins at least one additional unit with strictly positive probability. This deviation will increase the market clearing price by at most ε, and since i is high value when $s \in \underline{S}$, for $\varepsilon > 0$ sufficiently small this deviation is profitable. It follows that a low-revenue equilibrium cannot exist. □

The principles voiding low-revenue equilibria in multi-unit auctions are essentially the same as those applied to the single-unit second price auction.[15] If a strategy profile induces low revenue, bids must be sufficiently high to dissuade competition for units which are obtained, and bids for any marginal units must equal the reserve price. Because bidders occasionally have value profiles which fall below the reserve price, other bidders are occasionally in competition for any given bidder's demand. This competition ensures that bidders will never co-locate mass points, and hence there is a positive incentive to bid slightly above the reserve price when values are sufficiently large. It follows that with a relevant reserve price, per-unit revenues must be strictly above the reserve price with positive probability.

Theorem 2 (Existence of low-revenue equilibrium)**.** *If the reserve price $r > 0$ does not generate excess residual demand, there is a low-revenue equilibrium in the first rejected bid uniform price auction.*

Proof. Suppose that r does not generate excess residual demand. Then there is an agent $i^\star$ such that $\sum_{j \neq i^\star} \overline{m}^j(r) \leq Q$. Consider the strategy profile $(b^i)_{i=1}^n$, where $b_k^{i^\star}(s_{i^\star}) \in \{0, r\}$ for all units k and all signals $s_{i^\star}$, and

$$i \neq i^\star \implies b_k^i\left(s_i\right) = \begin{cases} 1 & \text{if } \mathbb{E}_{s_{-i}}\left[v_k^i\left(s_i, s_{-i}\right) \middle| \, s_i \right] \geq r, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sum_{j=2}^n \overline{m}^j(r) \leq Q$ and $b_k^{i^\star} \in \{0, r\}$ the bids of agents $j \neq i^\star$ never set the market clearing price; then since bidder $i^\star$'s bid is either r or zero, the market clearing price is $p^\star = r$. Then any bidder $j \neq i^\star$ receives her full stated demand (all units for which she bids 1) regardless of opponent signal realizations, and there is

[15] In the case of multiple units truthful reporting typically fails because an agent's supramarginal bids occasionally determine the market price. Then the nonexistence of low-revenue equilibrium cannot be demonstrated as a consequence of truthful reporting, but follows instead from the bisection principles that underlied truthful reporting in the single unit auction.

no information content in winning any number of units. Bidder $j \neq i^\star$ receives each unit she unconditionally values above r at a price of r, and her strategy is a best response.

Constructing bidder $i^\star$'s strategy is more delicate, since there is potentially information content in her allocation. Because bidders $j \neq 1$ are submitting bids that are either 0 or 1, given any bid vector $b \in \{0, r\}^Q$ bidder $i^\star$ cannot affect her resulting allocation unless she bids 1 (weakly above her value) or 0 (sacrificing the unit). Then it is a best response for her to submit a bid vector such that her bid for any unit is r if her expected value, conditional on winning this unit, is weakly above r, and zero otherwise. □

Corollary 2. *The first rejected bid uniform price auction with $n = 2$ bidders admits a low-revenue equilibrium.*

Putting together Theorem 1 and Theorem 2 gives a complete characterization of when low-revenue equilibria exist.

Proposition 1. *The first rejected bid uniform price auction with reserve price $r > 0$ admits a low revenue equilibrium if and only if r does not generate excess residual demand.*

Given the relative generality of our model of the first rejected bid uniform price auction, Proposition 1 suggests that bootstrapping competition by implementing a reserve price can be helpful in a wide range of circumstances.

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