This document contains a few small proofs, hopefully explained reasonably well. Econ 11 does not require you to be able to prove things. However, for exposition's sake I find it helpful to do something a little more than just state results. If you aren't into proofs, don't worry; if they help you, then good.

Economics (or, why Econ 11?)

Think of "economics" for a moment. What comes to mind?

There is no one particular reason any of us are studying economics; some are interested in finance, some are interested in central banks and money, and some are interested in welfare and policy.[1] The questions that draw many people to economics in the first place are those of very large scale: what happens when the Fed changes interest rates? when taxes are changed? when Facebook finally IPOs? In Econ 1 and Econ 2, you learned some tools for analyzing these issues in very broad strokes.

Econ 11 exists to uncover at least some part of why the tools you learned work. Although logic, reason, and intuition can guide a lot of what we say about large-scale, macroeconomic issues, it is difficult to speak precisely about any outcome without the addition of some mathematical formalism. In particular, what exactly is an economy?

At its simplest level, an economy is a collection of *agents* — be they single individuals, consumers, firms, children on a playground, or for that matter a colony of ants — who interact in some way. In the course of this class, we are going to figure out what it seems reasonable for individuals to do; slowly (and over the rest of your career in economics) we will see the implications for our predictions of behavior in the macroeconomy. That is, once we have an idea of what very simple groups do together, we will have some guidance regarding what we should see with much larger groups.

We should remember throughout that economics is as much an art as a science (if not more), and everything we will consider in this class is a gross simplification of the real world. Economists are of the view that it is better to obtain simple, possibly inaccurate results iteratively than it is to work for centuries on the one true model of everything; if we're being charitable, we can compare this to the movement from Newtonian mechanics to relativity to quantum mechanics. In this timeline, it's unclear where modern economics falls, but we are certainly still applying gross simplifications.

We will begin by considering the behavior of an idealized agent who acts according to very simple rules (rationality) which often turn out to be easy to analyze. We will uncover a fairly rich class of results that help support what we already "know" about economics. From there, it's up to the class syllabus.

Choice and preference

The fundamental issue to this point is that, while we have a set of agents together in an economy, they are completely inert. We need to put some context on what these agents should do. The most general formulation available is that agents make *choices*. We will, for the duration of this class, assume that agents are choosing from among goods (ice cream, water, Ferraris) and services (tax preparation, maid service); more generally, agents could just as well be choosing whether or not to punch each other in the face.

We begin by defining a *set of alternatives*, the set from which things may be chosen. If a consumer is trying to choose an apple or an orange, this might be

$$\{\text{apple}, \text{orange}\}.$$

Of course, this is fairly generic. Not only can an agent choose a commodity, she may choose different

[1] If we're being honest, some are interested in the fact that it's a major and they needed one; if that's you, don't worry.

quantities of this commodity, or some amount of commodity A and some amount of commodity B. To extend the above example, maybe the set of alternatives is

$$\{(1 \text{ apple}, 1 \text{ orange}), (2 \text{ apples}, 0 \text{ oranges}), (0 \text{ apples}, 2 \text{ oranges})\}.$$

We bundle together everything which is feasibly available into a single set, and the agent makes a selection.

As a rule, we will consider *commodity bundles*, not single commodities; that is, instead of picking "an apple," we think of the agent as picking "one apple and no oranges." In this way, we can represent choices by vectors of real numbers; if there are ℓ types of commodity — apples, oranges, etc. — we can represent a choice as a vector in $\mathbb{R}^\ell$, where the number in the i^{th} position indicates the number of commodity i selected. When convenient, we will drop the explicit tie of a dimension to a particular commodity and simply operate in a slightly more abstract world where there are nameless commodities available in certain quantities.

In particular, economics doesn't care if $(1, 0)$ is one apple and no oranges, or one orange and no apples, or one bushel of apples and no monkeys. Although the meaning of $(1, 0)$ will certainly affect your choice from the set of alternatives, this will be encoded in the next object we will look at.

From a set of alternatives, an agent makes a choice; we represent this as a *choice function*, generally called C. In a limited sense[2] the choice function tells us what would be chosen from a certain set of alternatives. As an example, suppose that the set of alternatives includes bundles $\{A, B, C\}$. The choice function might specify

$$\begin{aligned} C(\{A\}) &= A, & C(\{A,B\}) &= B, \\ C(\{B\}) &= B, & C(\{B,C\}) &= B, \\ C(\{C\}) &= C, & C(\{A,C\}) &= A, \\ & & C(\{A,B,C\}) &= B, \end{aligned}$$

That is, the choice function specifies what would be chosen from any set of available alternatives that might exist.

The beauty of choice is that it is eminently measurable: put someone into a store where options A and B are available, then watch what they do. The item they choose indicates the output of the choice function on that set of alternatives. Unfortunately, the choice function is not terribly compact in its notation, and analytically is not a good object to work with.

Preference relations

To help smooth the situation, we consider *preference relations*. A preference relation is denoted by $\succeq$ ($\preceq$) and means, in intuitive terms,

$$A \succeq B: \quad \text{“}A \text{ is at least as good as } B\text{,” or “}B \text{ is not preferred to } A\text{.”}.$$

We can use this to define two other symbols,

$$\begin{aligned} A \succeq B \text{ and } B \not\succeq A \quad &\Longrightarrow \quad A \succ B: \text{“}A \text{ is better than } B\text{,”} \\ A \succeq B \text{ and } B \succeq A \quad &\Longrightarrow \quad A \sim B: \text{“The agent is indifferent between } A \text{ and } B\text{.”} \end{aligned}$$

We refer to $\succeq$ as *weak preference*, $\succ$ as *strict preference*, and $\sim$ as *indifference*.

Intuitively, if $A \succ B$, then when A and B are both available you will certainly not choose B: A is available and you like it more.[3] If $A \sim B$, when A and B are the only two items available it is unclear which will be

[2] The generalization adds technicalities but no particular intuition

[3] We don't say "You will choose A," since perhaps some C is also available which is preferred to both A and B.

chosen: you are literally and completely incapable of making a judgment between one and the other, so you may as well pick A, or B.

We will always make the following assumptions:

- $\succeq$ is *complete*: for any two alternatives A and B, either $A \succeq B$ or $B \succeq A$, or both. This is identical to the definition given in lecture involving strict preference and indifference (as a quick exercise, check it!).

  Solution: In lecture, we were told that a preference relation was complete if for all A and B, either $A \succ B$ or $B \succ A$ or $A \sim B$. If $A \succ B$ or $A \sim B$, then $A \succeq B$; if $B \succ A$ or $B \sim A$, then $B \succeq A$. Since all pairs of good may be compared with strict preference (or are indifferent), then all pairs may be compared with weak preference.

  Now, assume that for all A and B, either $A \succeq B$ or $B \succeq A$ or both. If both, then $A \sim B$; if the former but not the latter, then $A \succ B$; if the latter but not the former, then $B \succ A$. Thus all pairs may be compared with strict preference (or indifference).

  It follows that the two definitions are equivalent. □

- $\succeq$ is *transitive*: if $A \succeq B$ and $B \succeq C$, then $A \succeq C$. *It is more difficult than the previous suggestion, but you can check that this implies that $A \succ B$ and $B \succeq C$ imply $A \succ C$.*

  Solution: Suppose, in the above case, that $A \not\succ C$. Since transitivity tells us that $A \succeq C$, it must be that $A \sim C$, which requires $C \succeq A$. Then we have $B \succeq C$ and $C \succeq A$, so $B \succeq A$; this contradicts the assumption that $A \succ B$, so it must be that $A \succ C$. □

- $\succeq$ is *continuous*: if $A \succ B$ and A' and B' are close to A and B, respectively, then $A' \succ B'$. This does have a more formal definition, but it is not worth going into.

A preference relation which is both complete and transitive is referred to as *rational*.

Where choice functions express what is chosen from particular sets of alternatives, and are useful because of their explicit relationship to the real-world observability of people choosing bundles from those available, preference relations are more abstract. They provide a complete reference for what an agent might do under various conditions. As an example, suppose there is one commodity in the world: water; we are interested in how much water an agent might consume. Now, keep in mind that the agent could consume 0 gallons or 1 gallon. Or 1.5 gallons, or π gallons, or $\frac{503}{71}$ gallons. There are an infinite number of consumption possibilities! There is no way we could ever observe all of an agent's choices from every subset of alternatives; preferences, on the other hand, let us express an agent's desires without requiring the underlying observability. In this sense they are more abstract and less practical, but for most mathematical purposes they are a vastly superior tool.

Practice problems

(a) Suppose that there are three alternatives, $\{A, B, C\}$. We know that $A \succ B$, $B \preceq C$, and $C \succ A$. Suppose that $\succeq$ is transitive. Do we know whether or not $B \sim C$?

Solution: Yes; in particular, $B \prec C$. We know that $B \sim C$ when $B \succeq C$ and $C \succeq B$. By assumption, $C \succeq B$. Now suppose that $B \succeq C$; since $C \succ A$ and $\succeq$ is transitive, this implies $B \succ A$ which is not the case. Hence $B \not\succeq C$, and so $B \prec C$. □

(b) Use the same setup above; introduce a new alternative D such that $C \succeq D \succ A$. If we assume that $\succeq$ is transitive, is $\succeq$ rational?

Solution: Yes. Recall that a preference relation is rational if it is both complete and transitive. Since we have assumed that $\succeq$ is transitive, we only need to check that it is complete. We know

	A	B	C	D
A	$\sim$	$\succ$	$\prec$	$\prec$
B		$\sim$	$\preceq$	???
C			$\sim$	$\succeq$
D				$\sim$

Since completeness requires that we can compare any two elements, we are set except for the relationship between B and D. However, we know $D \succ A \succ B$; since $\succeq$ is transitive, this implies that $D \succ B$. Hence we can replace ??? above with $\succ$, and we know the preference relationship between D and B. We can compare any two alternatives, so $\succeq$ is complete. □

Utility

Although considering preference relations is an advance over choice functions, they are by their very nature still equally unwieldy to apply in any useful way. However, under the assumptions that we stated — rationality (completeness and transitivity) and continuity — we can make a useful transformation: we can consider utility functions.

A *utility function* u is a map from the set of alternatives to the real numbers such that, for any two alternatives A and B,

$$u(A) \geq u(B) \iff A \succeq B.$$

That is, utility transforms consumption bundles to numbers such that a larger number represents a more-preferred item. This is extremely useful since we have many useful tools for dealing with real numbers (algebra, calculus, etc.); when considering choices in terms of utility, we can make much simpler inferences than we could if we were dealing with a pure preference relation.

Exercise: *show that the above definition of utility (in relation to preferences) implies that* $u(A) > u(B) \iff A \succ B$.

Solution: Suppose that $u(A) > u(B)$; then $u(B) < u(A)$, so it cannot be that $B \succeq A$. By completeness, we must have $B \succeq A$ or $A \succeq B$, so $A \succeq B$. With $A \succeq B$ and $B \not\succeq A$, we know that $A \succ B$.

Now suppose that $A \succ B$; then $B \not\succeq A$, so it cannot be that $u(B) \geq u(A)$. Hence $u(A) > u(B)$. □

As mentioned above, we will generally consider consumption bundles to be vectors in $\mathbb{R}^\ell$, expressing the amount of consumption in each of the ℓ available commodities. In a two-commodity economy, we will often refer to one commodity as x and the other as y;[4] with more than two commodities, we usually refer to goods x_1, x_2, ..., x_ℓ.

For an incredibly simple example, consider my utility for hamburgers. Experience tells me that one hambuger is better than no hamburgers,[5] two is better than one, but three or more is worse than consuming none. If we denote my set of alternatives simply by the number of hamburgers consumed, $H = \{0, 1, 2, 3, \ldots\}$, we have

$$(2) \succ (1) \succ (0) \succ (3) \succeq (4) \succeq \ldots$$

In utility space, this is

$$u(2) > u(1) > u(0) > u(3) \geq u(4) \geq \ldots$$

[4]There are of course exceptions; in particular, when dealing with money as a commodity, we usually denote it m.

[5]Let's say I'm at a decent restaurant.

Nonspecificity

In the above example, notice that we simply said that one alternative yielded more utility than another: we did not consider specific values. Let's assume now that

$$\begin{aligned} u(0) &= 0, & u(1) &= 1, \\ u(2) &= 2, & u(3) &= -1, \\ u(4) &= -1, & &\dots \end{aligned}$$

This satisfies the utility relation we stated above. But consider an alternative utility function v, where

$$\begin{aligned} u(0) &= e, & u(1) &= \pi, \\ u(2) &= 10, & u(3) &= -\frac{50}{7}, \\ u(4) &= -\frac{50}{7}, & &\dots \end{aligned}$$

Notice that v also satisfies the utility relation we stated above! This gets at a deeper issue: utility is *just a number*, it is not an absolute.[6] In particular, all that matters with regard to utility is whether the utility from one alternative is bigger than another. Utility is a unitless quantity.

Perhaps think about it this way: ask a five-year-old how much they like ice cream and watch them spread their arms and say "Thiiiiiiis much!" How much is that, really? When they're six and they've grown, does it mean that they like ice cream more since their arms are longer? Utility considers how far you spread your arms to answer this question, relative to other consumption goods; but it is otherwise completely meaningless.

This is captured nicely in the following theorem.

Theorem. *Suppose that u is a utility function representing $\succeq$, and f is a strictly-increasing function.*[7] *Then $v = f \circ u$ is a utility function representing $\succeq$.*

Proof. We must show that $v(A) \geq v(B) \iff A \succeq B$.

First, suppose that $A \succeq B$. Then $u(A) \geq u(B)$. Since f is a strictly-increasing function, this means that $f(u(A)) \geq f(u(B))$, or $(f \circ u)(A) \geq (f \circ u)(B)$. Then $v(A) \geq v(B)$.

Now suppose that $v(A) \geq v(B)$. Then $(f \circ u)(A) \geq (f \circ u)(B)$, or $f(u(A)) \geq f(u(B))$. Since f is strictly increasing, it follows that $u(A) \geq u(B)$. As u represents $\succeq$, this implies that $A \succeq B$.

Thus we know that $v(A) \geq v(B) \iff A \succeq B$, so v is a utility function representing $\succeq$. □

In economics, a strictly increasing function is referred to as *monotonic*,[8] so we will refer to this theorem as "taking a monotonic transformation."

Strength of preference

As an aside, someone asked in question what this means about expressing the *magnitude* of preference. That is, if you can choose between one hamburger or two, how can we consider that two hamburgers is *much* better than one versus two hamburgers being only a little better than one? The short answer is, not with utility; or, not in the context issued, anyway. Since economists are concerned only with what is chosen, *why*

[6] This is actually a fairly deep and modern consideration. Up through the mid-1800s (at least) some philosophers were convinced that the number had deep and resonant meaning.

[7] Math review: a function f is *strictly increasing* if $x > y$ if and only if $f(x) > f(y)$.

[8] The reason I say "in economics" here is that the actual definition of a monotonic function is somewhat broader; so far as economists are concerned, though, strictly increasing is the only interesting case.

it is chosen is not a primary issue. In particular, regardless of how much better two hamburgers are than one, I would still choose two over one.

This is a less than satisfying answer since, as we all know intuitively and in our daily lives, there *are* magnitudes of preferences. What is worth considering, though, is does the magnitude of preference matter in an economy with only hamburgers? In a sense, magnitude captures how much you'd be willing to give up in order to obtain the second hamburger; with just hamburgers, there's nothing to give up! In a world with another commodity — say, money — you can express the magnitude of your preference through your willingness to part with dollars to get the second hamburger.

To make this concrete, suppose your utility for hamburgers is

$$u_H(y) = \begin{cases} y^2 & \text{if } y \le 2, \\ -1 & \text{otherwise.} \end{cases}$$

Your utility for hamburgers and money is $u(m, y) = m + u_H(y)$. We can, in a way, express the magnitude of your preference for two hamburgers over one by looking at the amount of money that leaves you indifferent between two hamburgers and one. In particular, if m_1 and m_2 are amounts of money, for you to be indifferent between two hamburgers and one it must be that

$$u(m_1, 1) = u(m_2, 2) \quad \Longrightarrow m_1 + 1 = m_2 + 4 \quad \Longrightarrow m_1 - m_2 = 3.$$

So you are willing to give up 3 dollars for the second hamburger.

Now consider that your utility for hamburgers is

$$v_H(y) = \begin{cases} \sqrt{y} & \text{if } y \le 2, \\ -\frac{1}{2} & \text{otherwise.} \end{cases}$$

Exercise: *does v_H represent the same preferences for hamburgers alone (remember that hamburgers come in whole amounts, not fractions)? Does $v(m, y) = m + v_H(y)$ represent the same preferences for hamburgers and money? Which utility function represents "stronger" preferences for the second hamburger?*

Solution: you can check the first part directly, but it's faster to apply the theorem we stated earlier. Let f be an increasing function defined by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < 0, \\ x^{\frac{1}{4}} & \text{otherwise.} \end{cases}$$

Then we can see that $v_H = f \circ u_H$, so v_H also represents my preferences for hamburgers.

For the second part, we will look for a counterexample. We know that

$$u(1, 1) = 2 < 4 = u(0, 2),$$

so it must be that $(1, 1) \prec (0, 2)$ according to the first preference relation. However,

$$v(1, 1) = 2 > \sqrt{2} = v(0, 2),$$

so $(0, 2) \prec (1, 1)$ according to the second preference relation. Since these don't match, u and v cannot represent the same preferences.

Lastly, we check how much money I am willing to give up to obtain the second hamburger over the first. As before, we check

$$v(m_1, 1) = v(m_2, 2) \quad \Longrightarrow \quad m_1 + 1 = m_2 + \sqrt{2} \quad \Longrightarrow \quad m_1 - m_2 = \sqrt{2} - 1 \approx 0.414.$$

Then according to the second preference relation, I am only willing to give up 0.414 units of money for the second hamburger, while according to the first I am willing to give up 3 units of money for the second hamburger. In an intuitive sense, then, the first utility function u represents stronger preferences for hamburgers.
□

Indifference curves

The last concept we will concern ourselves with is that of an *indifference curve*, or *isoutility curve*.[9] An indifference curve is a set of commodity bundles all yielding the same utility (hence, "isoutility curve"). In particular, the indifference curve through a point $(x_1, \ldots, x_\ell)$ is defined by

$$U = \{(x_1', \ldots, x_\ell') : u(x_1, \ldots, x_\ell) = u(x_1', \ldots, x_\ell')\}.$$

By definition, this set is equivalent to

$$U = \{(x_1', \ldots, x_\ell') : (x_1, \ldots, x_\ell) \sim (x_1', \ldots, x_\ell')\}.$$

Now, dealing with such high-dimensionality concepts is messy, so we will often constrain ourselves to the case of two commodities, x and y. In this special case, we can usually (in Econ 11, anyway) consider an indifference curve to be a function; the indifference curve through (x, y) can be expressed as

$$f(x') = y' \text{ such that } u(x', y') = u(x, y).$$

Recognizing that this is incredibly abstract, this will be made more concrete in the next section.

Common utility functions

There are a few utility functions that we will see over and over again in class. It is good to get a handle on them.

Linear utility

A utility function is *linear* if it may be represented as

$$u(x_1, \ldots, x_\ell) = a_1 x_1 + \ldots + a_\ell x_\ell + c,$$

where a_1 through a_ℓ are constants, and so is c. Now, remembering that only the comparison between utilities matters (that we can take a monotonic transformation) the additive constant c is unimportant; hence utility may be represented as

$$u(x_1, \ldots, x_\ell) = a_1 x_1 + \ldots + a_\ell x_\ell.$$

In the two-commodity case, this is

$$u(x, y) = a_x x + a_y y.$$

Fixing a level of utility $\overline{u}$, we can find an indifference curve by solving for y in the equation

$$\overline{u} = a_x x + a_y y \quad \Longrightarrow \quad y = \frac{\overline{u} - a_x x}{a_y}.$$

Notice that, taking any x, the y returned by this function yields the same utility (by construction; this is the equation we solved). Hence the agent is indifferent across all points on this curve.

[9] If you remember isotherms from high school physics/chemistry, the analogy is straightforward.

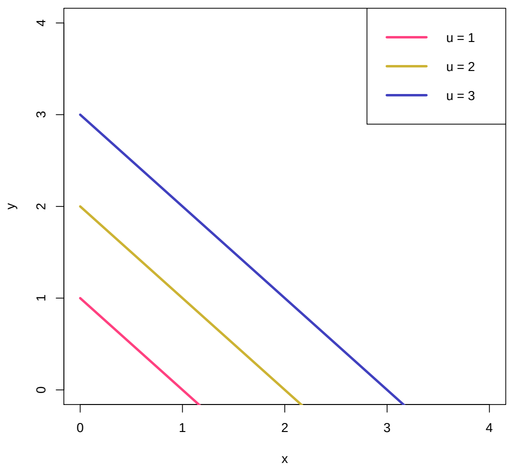

Figure 1: indifference curves with linear utility and $a_x = a_y = 1$.

Cobb-Douglas utility

A utility function is *Cobb-Douglas* if it may be represented as *The actual definition is somewhat more general, but this is a needless technicality.*

$$u(x, y) = x^{\alpha} y^{\beta},$$

where $\alpha > 0$, $\beta > 0$, and $\alpha + \beta = 1$.

To compute indifference curves, we again solve (fixing a level of utility $\overline{u}$)

$$\overline{u} = x^{\alpha} y^{\beta} \quad \Longrightarrow \quad y = (\overline{u} x^{-\alpha})^{\frac{1}{\beta}}.$$

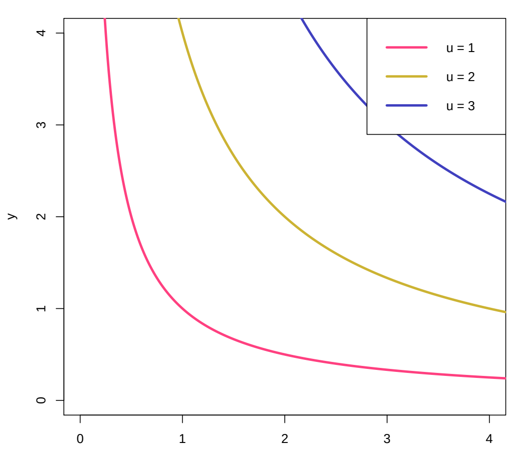

Figure 2: indifference curves with Cobb-Douglas utility and $\alpha = \beta = \frac{1}{2}$.

CES utility

A utility function is *CES*[10] if it may be represented as

$$u(x_1, \ldots, x_\ell) = \sum_{i=1}^{\ell} \frac{x_i^\gamma}{\gamma}$$

where $\gamma \leq 1$ and $\gamma \neq 0$. In the two-dimensional case, this is

$$u(x, y) = \frac{x^\gamma}{\gamma} + \frac{y^\gamma}{\gamma}.$$

To compute indifference curves, we again solve (fixing a level of utility $\overline{u}$)

$$\overline{u} = \frac{x^\gamma}{\gamma} + \frac{y^\gamma}{\gamma} \qquad \Longrightarrow \qquad y = (\gamma\overline{u} - x^\gamma)^{\frac{1}{\gamma}}.$$

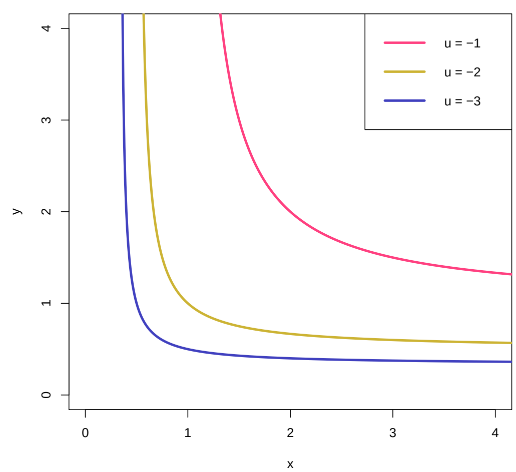

Figure 3: indifference curves with CES utility and $\gamma = -1$.

Quasilinear utility

A utility function is *quasilinear* if it may be represented as

$$u(x_1, \ldots, x_\ell) = x_1 + v(x_2, \ldots, x_\ell).$$

for some function v. In the two-dimensional case, this is

$$u(x, y) = x + v(y).$$

Computing indifference is a little trickier here; in particular, there is no exact analytical solution since v is an arbitrary function. If we assume v is invertible, we fix a level of utility $\overline{u}$ and compute

$$\overline{u} = x + v(y) \qquad \Longrightarrow \qquad y = v^{-1}(\overline{u} - x).$$

[10]Constant elasticity of substitution.

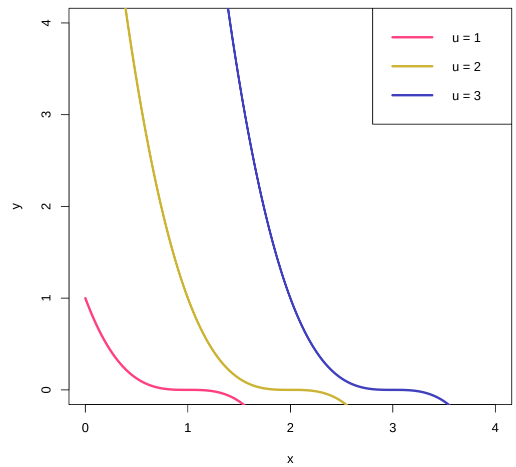

Figure 4: indifference curves with quasilinear utility and $v(y) = \sqrt[3]{y}$.

Leontief utility

Lastly, a utility function is Leontief if it may be represented as

$$u(x_1, \ldots, x_\ell) = \min\{a_1 x_1, \ldots, a_\ell x_\ell\},$$

where a_1 through a_ℓ are constants. In the two-dimensional case, this is

$$u(x, y) = \min\{a_x x, a_y y\}.$$

If this utility function seems silly, consider the canonical example of utility for shoes; here, x is the number of left shoes you have and y is the number of right shoes you have, and $a_x = a_y = 1$. A right shoe is useless without a left shoe, and vice-versa. It is reasonable to think that your utility for shoes is simply the total number of pairs of shoes you have, or the minimum of the number of left and right shoes you have.

Indifference curves are also tricky in this case: notice that, fixing a level of utility $\overline{u}$, if $a_x x < \overline{u}$ there is *no quantity of* y such that $u(x, y) = \overline{u}$. Further, if $\overline{u} = a_x x$, any amount of y such that $a_y y \geq \overline{u}$ will yield the same amount of utility. With this in mind, the indifference curve can be somewhat loosely expressed as

$$y = \begin{cases} \frac{\overline{u}}{a_y} & \text{if } a_x x > \overline{u}, \\ \left[\frac{\overline{u}}{a_y}, +\infty\right) & \text{if } a_x x = \overline{u}, \\ \emptyset \text{ (nothing)} & \text{otherwise.} \end{cases}$$

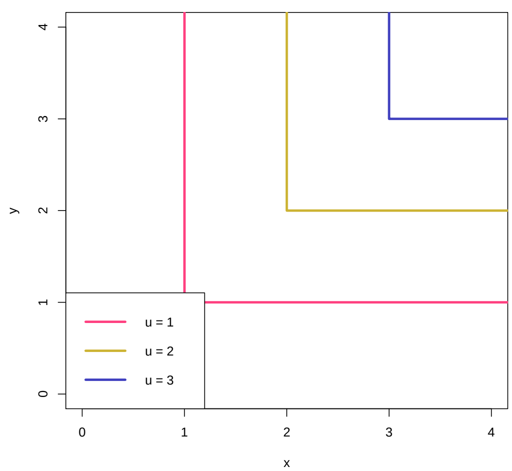

Figure 5: indifference curves with Leontief utility and $a_x = a_y = 1$.