Equilibrium and Approximation in Auction-Like Models

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Abstract

I define auction-like models with continuum actions to satisfy a number of conditions which are met in common auction contexts, when action spaces are appropriately constrained. In these models, I prove that there exists a monotone pure-strategy Bayesian-Nash equilibrium. The proof of equilibrium existence constructs equilibrium as a limit of equilibria of nearby discretized models, which implies that utility-relevant observable functions of actions are converging in probability to the continuum-action limit. The conditions defining auction-like models are intuitive: first, bidder utility cannot be adversely affected by small upward deviations; second, with small deviations bidders can guarantee themselves nearly the utility at a limit of strategies in the limit of the deviated strategies; third, if one bidder is discontinuously worse off in a limit of strategies, some other bidder is occasionally better off. Under mild additional conditions, the action space constraints are irrelevant, and the limiting strategies are an equilibrium in the unconstrained continuum-action model. I use these results to prove the existence of pure-strategy equilibria in divisible-good pay-as-bid auctions with private information, heterogeneous agents, and generic decreasing value functions. Equilibrium approximation implies that the distribution of observed allocations and seller revenue in discrete auctions may be close to the distribution of these outcomes predicted by the divisible-good model.

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1 Introduction

When auction models are generalized to consider the sale of multiple units, pure-strategy equilibria frequently continue to exist (Reny, 2011; McAdams, 2003) but become intractible to compute (Hortaçsu and Kastl, 2012). As in other economic contexts, there is hope that continuum models can provide a useful path around this intractibility, however it has so far been unknown whether in the presence of private information a pure-strategy equilibrium even exists in many such models. In this paper I identify a number of conditions which are intuitively satisfied by many "auction-like" models,[1] and subject to these conditions I prove that a pure-strategy equilibrium exists and that observable outcomes of the continuum-action auction can approximate those in nearby discretized auctions. I show that the divisible-good pay-as-bid auction satisfies these conditions, implying that a pure-strategy equilibrium exists in this model, addressing a question that has been heretofore unresolved in the literature.

The proof of equilibrium existence is based in the observation that in many auctions, actions are equivalent to monotone functions. Within any monotone pure-strategy equilibrium, this implies that strategies are multidimensional monotone functions. For example, in a single-unit auction a "bid" is a mapping from the singleton set $\{0\}$ to the realized value of the bid; in the same context, mixed strategies can be modeled as pure strategies, where the CDF of the mixture is a monotone function. In multi-unit and divisible-good auctions, actions are typically decreasing functions across the quantity domain (in procurement auctions the direction of monotonicity is reversed, but monotonicity still holds).[2] Hence "continuum-action" in this context is more general than simply selecting values from a continuum: it is selecting maps from one continuum to another.

Actions in the continuum model are discretized by constraining their domain and range to a grid of points. In the discretization many traditional assumptions for equilibrium existence are satisfied—by finiteness all utility functions are continuous in action, and convergent sequences of actions are eventually constant—hence verifying the existence equilibrium in the discretized model is a comparatively simple task. When there is a monotone pure-strategy equilibrium in a refining sequence of discretized models, that actions are monotone functions enables a relatively straightforward construction of a limiting action profile. It then remains only to be shown that the limiting action profile is, in fact, an equilibrium of the continuum-action

[1] As a useful contrast, one of the simpler of these conditions is not satisfied by the divisible-good uniform-price auction, a continuum approximation of a common multi-unit auction mechanism. Interestingly, this condition fails only at the continuum limit and not for any multi-unit counterpart; this is discussed briefly in Section 5.

[2] In multi-unit auctions the decreasing bid function is traditionally modeled as a vector of decreasing bids. Since I assume that the domain of the functions-cum-actions is compact and convex, this is captured as a decreasing step function. Because I do not impose independent structure on how "bids" are extracted from actions, this construction is strictly more general.

model.

I show that the limiting strategies are mutual best responses by showing that utility converges in the limit. The constructed strategies are therefore relatively close to equilibrium strategies in the sequence of discretizations, providing a useful check on the potential empirical value of the continuum-action model. A further implication of this construction is that if a measurable outcome is utility-relevant—that is, if its nonconvergence implies nonconvergence of utility—then it is converging in probability to its distribution in the continuum-action model. For example, I show in examples that observed quantity allocations and seller revenue in common auctions are converging to their distributions in the continuum-action model.[3] Since empirical investigations of auctions frequently address questions of efficiency and revenue, these results provide a sound justification for application of the continuum-action model.

With these results I am able to prove the existence of pure-strategy Bayesian-Nash equilibria in divisible-good pay-as-bid auctions. Under the presence of private information, this question has been heretofore unanswered. Together with the equilibrium approximation results, this suggests that divisible-good models may present a useful strategy for analyzing multi-unit auctions with large numbers of units. Generally, the results in this paper present a unified approach to equilibrium existence across auction-like contexts. With independent, private values, I show that the conditions defining auction-like models are satisfied by first-price auctions as well as by divisible-good pay-as-bid auctions, and the intuition behind each of the satisfaction of the conditions is essentially the same in each case.[4] The technical conditions placed on auction-like models are related to those given by the thread of equilibrium existence results begun by Reny (1999). In some cases they are stronger—for example, better-reply security is replaced with local better-reply security—while in other cases they are weaker—for example, quasisupermodularity is not essential, and continuity can be relaxed to uniform upper semicontinuity.[5]

Auction-like models are defined by the satisfaction of a number of conditions, some more innocuous and others more essential. The relatively innocuous assumptions are representative of technical constraints on the analysis and do not necessarily correspond directly to practical concerns; for example, utility functions must be well-defined on sets that include upper discretizations of actions in the continuum model (Condition 9). The more meaningful conditions are set forth in Sections 2.2 and 2.3, and are discussed at length in the remainder of the paper. A brief synopsis

[3]The interchanging of "probability" and "distribution" here is valid, since I prove convergence in probability, which implies convergence in distribution.

[4]Of course, there is little theoretical value in establishing the existence of pure-strategy equilibrium in single-unit first-price auctions with independent private values. The purpose of the analysis of this auction is to present the conditions defining auction-like models in a familiar and well-understood context.

[5]Naturally some of the conditions must be weaker, otherwise Reny would be sufficient to show equilibrium existence in divisible-good pay-as-bid auctions.

follows here.

First, models must be uniformly upper semicontinuous in an agent's own action: an infinitesimal increase in bid[6] cannot yield a discontinuous drop in utility, nor can an infinitesimal decrease in bid yield a discontinuous jump in utility.[7] To satisfy this condition I must allow for type-dependent action spaces. For example, in a first-price auction, under the assumption that bidders bid below their true valuations, this condition is easily verified: a slight increase in bid can yield a discontinuous increase in the probability of winning the object, but it cannot yield a discontinuous increase in the payment for the object (conditional on winning).[8]

Second, given a convergent sequence of their opponents' strategies agents must be able to "lock in" limiting utility. If, given a sequence of her opponents' bids, an agent's utility is better at the limit than in the limit, there is a nearby action that allows her to guarantee herself, in the limit, most of her utility at the limit in the other action. In practical auctions, this generally manifests as tiebreaking occurring at the limit, while being unnecessary near the limit. An agent can break a tie in her favor by increasing her bid slightly, at nominal cost. This ties very clearly to the convergence of utility in a sequence of equilibria, as an upward jump in utility at the limit would be nearly obtainable by small deviations near the limit.

Third, models must involve a shared surplus. If, given a sequence of all agents' strategies, one agent's utility is worse at the limit than in the limit, there is another agent whose utility is better, with positive probability, at the limit than in the limit. As above, an agent's utility dropping at the limit is commonly related to tiebreaking; if one agent begins losing a tie, then another agent begins winning a tie. This represents a weaking of the common assumption of reciprocal upper semicontinuity, in that no player's interim utility needs to discretely improve; there merely needs to be a sequence of actions that generates strictly greater utility against some subset of opponents.[9] Some care must be taken to ensure that gains occur with positive probability, but the intuition is straightforward. Under uniform upper semicontinuity, this roughly implies that near the limit the agent can employ a small upward deviation for a discrete gain at minimal cost.

Because the proof of equilibrium existence leverages the results of Reny (2011), the constructed Bayesian-Nash equilibrium comprises best responses for almost all signal realizations of all agents. The proof that the constructed equilibrium implies best responses for all signal realizations of all agents involves a further condition: any

[6]Appropriate to a class of models described as "auction-like," I will use "action" and "bid" interchangably. Similarly, I interchangeably refer to "agents" and "bidders."

[7]All proofs go through equally well if the opposite, uniform lower semicontinuity, is satisfied; see Section 3.2. Generally, satisfaction of this criterion can be easily allowed to vary by agent, but it is more difficult to relax to type-dependent heterogeneity (unless the types are exogenously given, or satisfy simple conditions like threshold rules).

[8]Without type-dependent action spaces the first-price auction obviously violates this condition: if an agent is bidding above her value, a slight increase from a tie yields a discontinuous downward drop in utility.

[9]For a related condition, see Bagh (2010).

action in the frontier of an agent's type-dependent feasible action space has an action in its interior which generates almost as much utility; since I assume that utility is continuous and increasing in signal, this roughly corresponds to the type-dependent action spaces varying continuously with signal. A slightly stronger condition, that the utility generated by any infeasible action outside of the type-dependent action space can be approximated by an action in the interior of the agent's type-dependent action space, ensures that the type-dependent action space is irrelevant, and the constructed equilibrium is an equilibrium even in the unconstrained model.

The proof of existence then proceeds by constructing a sequence of equilibria in nearby models for which known existence results are simpler to apply. For example, in finite models—discretizations of the continuum-action model—utility functions are automatically continuous in actions; in other cases, restricting action spaces can obviate the need for tiebreaking altogether,[10] allowing for the application of standard results using quasisupermodularity.[11] From this sequence of equilibria a convergent limit is constructed; this is the first block of the proof. Establishing that this limit represents mutual best responses is the second nontrivial task.

Because equilibrium in the continuum-action model is built as a convergent limit of a sequence of equilibria in nearby models, it follows almost immediately that any observed outcome that is continuous in actions will be converging along the same path of equilibria. This can be leveraged to provide a useful empirical statement: the distribution of outcomes in the continuum-action model can approximate the distribution of observed outcomes in the "real world" model it is meant to approximate. I generalize the notion of continuity of observables slightly to utility-relevance, which requires that observables are at least as continuous as utility functions. Inasmuch as many auction observables are discontinuous while I show that utility is converging in the limit of strategies I construct, this presents significantly greater value than simple continuity.

I use this equilibrium existence result to establish the existence of a pure-strategy equilibrium in two auction models. In the first application, I show that first-price auctions for single, indivisible units admit pure-strategy Bayesian-Nash equilibria when agents have independent private values. This is not novel, but provides a useful service in illustrating the intuitive nature of the technical conditions I place on auction-like models. In the second application, I show that divisible-good pay-as-bid auctions with general value functions admit pure-strategy Bayesian-Nash equilibria. This has been an open question.

[10]In divisible-good auctions, for example, restricting attention to Lipschitz-continuous inverse demand functions implies that rationing rules are never employed, typically rendering payoffs continuous.

[11]In these results I restrict attention to the discretization method. With respect to the more general case, equilibrium in the model of interest can be built in the same way from an appropriate selection of equilibria in the nearby games, and all other results go through. This, however, breaks the intuition of continuum-action models as approximations of discrete applications.

1.1 Related literature

This paper follows neatly from two threads of equilibrium existence literature begun by Reny (1999) and Athey (2001). The former establishes (potentially mixed-strategy) equilibrium existence in models with discontinuous payoffs, and the latter looks at the same question in models with private information and continuous payoffs. McAdams (2003) extends Athey's result to include multidimensional private information, and Van Zandt and Vives (2007) and Reny (2011) generalize to the case of arbitrary lattices. These results cannot be directly applied because, as is common in auction models, payoff discontinuities cannot be ruled out ex ante. Reny (1999) allows for discontinuous utility functions, but does not permit private information; his results have been extended by McLennan et al. (2011) and Barelli and Meneghel (2013). The results of Reny (2011) are directly employed to construct the desired best responses, but for the majority of the results McAdams (2003) are sufficient; applying Reny (2011) allows Corollary 1, which proves the existence of symmetric equilibria in symmetric models.

The approach of establishing equilibrium as a limit of nearby discretized equilibria has been used by, among others, Simon (1987), Reny and Zamir (2004), Bagh (2010), and Kastl (2012). In contrast to my pure-strategy existence result under private information, Simon (1987) establishes existence in mixed strategies, without private information.[12] The limiting approach of Reny and Zamir (2004) uses a similar approach to prove the existence of pure-strategy equilibria in first-price auctions; like my results here, it relies on convergence of utility, but unlike my results actions are point bids rather than generic monotone functions. Kastl (2012) provides equilibrium in distributional strategies with finite bid points, and uses this to suggest the same when bids can be arbitrary nonincreasing functions of quantity.

The condition most directly related to the ability of my conditions to extend existence results to the divisible-good pay-as-bid auction with private information is a weakened form of reciprocal upper semicontinuity. Similar conditions have been examined by Bagh and Jofre (2006), Bagh (2010), Allison and Lepore (2014), and He and Yannelis (2016). Bagh and Jofre (2006) examines weak reciprocal upper semicontinuity, which is not necessarily satisfied by pay-as-bid auctions; Condition 5A requires only that most of the limiting utility can be obtained, and not that it can be dominated. Bagh (2010) employs variational convergence, which invokes dominating sequences of actions; Condition 5B requires only that the dominating sequence of actions dominate the original sequence, and makes no claims at the limit. Lastly, Allison and Lepore (2014) and He and Yannelis (2016) introduce (random) disjoint payoff matching, again requiring dominance in the limit. None of these conditions

[12] In later work, McAdams (2006) shows that these mixed strategies can be rendered into monotone pure strategies without affecting best-responsiveness. Reny (1999) shows a related result, that in a particular multi-unit auction model the mixed strategies predicted are in fact pure strategies. Other results regarding equilibrium in mixed or distributional strategies include Milgrom and Weber (1985), Kastl (2012), and He and Yannelis (2016).

is obviously satisfied in divisible-good pay-as-bid auctions.

With regard to multi-unit auctions, relatively little is known about bidder behavior in the presence of private information. Beyond the apparent theoretical difficulty of computing fully general revenue and efficiency rankings, progress in the analysis of parameterized models has been hampered by the inability to efficiently compute equilibrium strategies in the case where goods, as in practice, are imperfectly divisible. Meaningful results have been obtained in certain settings—see, e.g., Engelbrecht-Wiggans and Kahn (2002), Ausubel et al. (2014), and Lotfi and Sarkar (2015)—but the general state of the art is best captured by Hortaçsu and Kastl (2012), who state, "Unfortunately, computing equilibrium strategies in (asymmetric) discriminatory multi-unit auctions is still an open question."[13] Häfner (2015) demonstrates the existence of an equilibrium in distributional strategies in a pay-as-bid auction with constrained bids, but does not obtain a pure-strategy existence result.

Where discrete problems appear intractible, continuous approximations may offer sound and available economic insights. For example, the literature on single-unit auctions frequently employs the assumption that the set of available prices is dense. In the case of multi-unit auctions, bids may be approximated as objects determined on a dense domain of quantities, as well; there is no counterpart to this possibility in single-unit auctions, or even in combinatorial auctions. Wilson (1979) was the first to apply this approximation method in the context of multi-unit auctions, and this approximation has been used to establish results for parameterized models such as Back and Zender (1993), Wang and Zender (2002), Ausubel et al. (2014), and Pycia and Woodward (2016), but in the general case it has not even been known if an equilibrium exists. Without a sound basis for the existence of equilibrium strategies, it has been difficult to meaningfully apply the divisible-good model to policy debates.

This paper proceeds by defining the underlying model and its discretization, in Section 2; this Section also lays out the assumptions that define an auction-like model. In Section 3 the canonical two-bidder, first-price auction is defined as an auction-like model and the assumptions are verified for exposition. Section 4 lays out the main results of the model. Section 5 takes these results to the divisible-good pay-as-bid auction model and proves the existence of pure-strategy equilibria as well as equilibrium approximation, and Section 6 concludes.

2 Model

Denote a model by $\mathcal{M} = (n, u, X, A, F, Z)$. There is a set of agents $i \in \{1, \ldots, n\}$, where each agent i has a bounded utility function $u^i : \hat{Y}^n \times (0, 1) \times \operatorname{Supp} Z \to \mathbb{R}_+$, and $u = (u^i)_{i=1}^n$. Agent i has private information $s_i \sim F^i$, $F = (F^j)_{j=1}^n$. For each

[13] There has been work in building approximate equilibria for multi-unit auctions; see, e.g., Armantier and Sbaï (2006), Armantier et al. (2008), and Armantier and Sbaï (2009).

agent, Supp $F^i = (0,1)$, and $s_i \perp\!\!\!\perp s_j$ for all $j \neq i$; F^i has no mass points. Because I assume that $u^i(y;\cdot)$ is continuous but make no assumptions on differentiability, it is without loss of generality to assume that $F^i = \mathcal{U}(0,1)$.[14] There is an independent and exogenous source of randomness $z \sim Z$, $z \perp\!\!\!\perp s_i$ for all i.

The tuple $X = (X_D, X_R)$ describes the domain and range of agents' actions, where X_D and X_R are compact and convex subsets of $\mathbb{R}$. Let $\hat{Y}_+$ and $\hat{Y}_-$ be the set of monotone increasing and decreasing, respectively, functions from X_D to X_R; $\hat{Y} \in \{\hat{Y}_+, \hat{Y}_-\}$.[15] For each agent i, the feasible action space A^i is type-dependent, $A^i : (0,1) \rightrightarrows \hat{Y}$, and given a signal $s_i \in (0,1)$ agent i's feasible action space is $A^i(s_i)$; $A = (A^i)_{i=1}^n$.[16] A strategy α^i for agent i is a mapping, $\alpha^i : (0,1) \to \hat{Y}$; a strategy is *feasible* if $\alpha^i(s_i) \in A^i(s_i)$ for all $s_i \in (0,1)$. A strategy profile $(\alpha^i)_{i=1}^n$ is feasible if α^i is feasible for all agents i.

Agent i's interim utility is given by U^i,

$$U^i\left(a_i, \alpha^{-i}; s_i\right) = \mathbb{E}_{s_{-i},z}\left[u^i\left(a_i, \alpha^{-i}\left(s_{-i}\right); s_i\right)\right].$$

For notational convenience, it is useful to denote agent i's expected utility over the exogenous randomness z as u_z^i,

$$u_z^i\left(a_i, a_{-i}; s_i\right) = \mathbb{E}_z\left[u^i\left(a_i, a_{-i}; s_i, z\right)\right].$$

Where no confusion exists, u_z^i will be termed *ex post utility*.[17]

Unless otherwise stated, all norms are L^1 on X_D, $||y|| = \int_{X_D} |y| dx$.

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[14] In the proofs in the Appendix, this manifests as interchanging measure and probability, or almost-everywhere and almost-surely. All results continue to go through with arbitrary distributions with bounded, convex, open support.

[15] The restriction that all functions in $\hat{Y}$ are monotone in the same direction is not essential, nor is the later assumption that all feasible actions are monotone in the same direction, provided that for each agent and signal realization it can be exogenously stipulated which direction monotonicity holds—all proofs would go through in this generalization, but the fact that feasible actions are a subset of $\hat{Y}$ implies that the model does not permit this relatively simple extension. Because many of the conditions laid out in Subsection 2.2 are in terms of functions in $\hat{Y}$, allowing for, say, increasing functions in $\hat{Y}$ in a model in which only decreasing functions make sense would require a translation function τ to map increasing functions to decreasing functions. This can be achieved without technical difficulty, but is unwieldy and yields no additional insight; see Section 3.2 for more on this.

[16] That agent i's feasible action space is type-dependent is not essential, and the standard assumption of homogeneous action spaces can be accounted for by assuming A^i is constant. Nonetheless, allowing for type-dependence simplifies the analysis of certain kinds of models, including divisible-good auctions. The proofs employed in Appendix B homogenize the action space by constructing auxiliary utility functions in which infeasible actions are never optimal; this could be achieved at the model level, but it is often more straightforward to state utility as a natural object and constrain actions.

[17] This model can be specified without randomness $z \sim Z$, taking the utility functions u_z^i as the true ex post utility functions; this does not affect any results. In some cases of interest, however, it is more straightforward to express the economic fundamentals underpinning the model using explicit randomness. See, for example, Section 5.

2.1 ε-discrete models

The proof of equilibrium existence proceeds from the examination of a sequence of approximations of the feasible action spaces A^i. In many cases it is straightforward to verify the necessary properties of the approximating action spaces if they represent discretizations of the model of interest. Given a *base model* $\mathcal{M} = (n, u, X, A, F, Z)$ and $\varepsilon = (\varepsilon_D, \varepsilon_R) \gg 0$, the ε*-discrete model* $\mathcal{M}^\varepsilon = (n, u, X, A^\varepsilon, F, Z)$ is derived from the base model, and has a modified feasible action space. Let $\hat{Y}^\varepsilon$ be the set of all monotone (in the direction of $\hat{Y}$) functions from $\boxed{X_D}^{\varepsilon_D}$ to $\boxed{X_R}^{\varepsilon_R}$, where

$$\boxed{X_D}^{\varepsilon_D} = \left\{ \left\lfloor \frac{x}{\varepsilon_D} \right\rfloor \varepsilon_D : x \in X_D \cup (X_D + \varepsilon_D) \right\},$$
$$\boxed{X_R}^{\varepsilon_R} = \left\{ \left\lfloor \frac{x}{\varepsilon_R} \right\rfloor \varepsilon_R : x \in X_R \cup (X_R + \varepsilon_R) \right\}.$$

For each agent i and signal $s_i \in (0,1)$, the feasible action space $A^{i,\varepsilon}(s_i)$ in $\mathcal{M}^\varepsilon$ is a subset of the available monotone functions, $A^{i,\varepsilon}(s_i) \subseteq \hat{Y}^\varepsilon$.

One natural case of interest is an equally-spaced grid, $\varepsilon_D = \varepsilon_R$, with discretized actions upper approximations of functions in A^i,

$$A^{i,\varepsilon}(s_i) = \left\{ y \in \hat{Y}^\varepsilon : \exists a \in A^i(s_i), y = \inf \left\{ y' \in \hat{Y}^\varepsilon : y' \geq a \right\} \right\}.$$

That is, $A^{i,\varepsilon}(s_i)$ is the set of closest approximations (from above) of functions in $A^i(s_i)$ by ε-step functions; because each $y \in \hat{Y}^\varepsilon$ is a map from a finite set to a finite set, $\hat{Y}^\varepsilon$ is finite and hence the infimum exists in $\hat{Y}^\varepsilon$.[18] As in the base model, $A^\varepsilon = (A^{i,\varepsilon})_{i=1}^n$. In other cases more genericity is necessary; conditions on $A^{i,\varepsilon}$ in these cases are given in Conditions 6A and 7A.

2.2 Auction-like models

Proofs are found in Appendix A.

In this Subsection I set forth the necessary conditions for a model to be "auction-like," in terms of interim utility; this form exposes the relation to the existence results of the thread of research begun by Reny (1999).[19] Subsection 2.3 gives sufficient assumptions on ex post utility, and Section 3 puts the assumptions in the well-understood context of single-unit, first-price auctions.

Condition 1 (Utility and actions)**.** *For each agent i, u^i is bounded, and continuous and increasing in signal. For all s_i, $A^i(s_i)$ is a complete lattice.*

[18] Here, by discreteness and finiteness of $\hat{Y}$, the infimum is the minimum; the use of infimum is for consistency with later results.

[19] The proof of equilibrium existence does not make use of the results in Reny (1999), but viewing the interim game of incomplete information as an ex ante game of complete information there is an evident connection between the results.

Condition 1 ensures well-behavedness of the underlying optimization problem (but not necessarily of the objective function) and of the techniques employed in proving equilibrium existence. Boundedness ensures the existence of convergent subsequences,[20] and continuity ensures that agents who are close in type should have elements in their best responses which are near to one another.

Condition 2 (Imitability)**.** *For each agent i, $s_i < s_i'$ implies $A^i(s_i) \subseteq A^i(s_i')$.*

A natural (but by no means unique) definition of A^i will be the set of actions which generate utility above some outside option. Since utility is increasing in signal, under this interpretation A^i is weakly increasing in set inclusion order. Higher-signal agents can always imitate lower-signal agents, but not vice-versa.

The next three conditions are properties of interim utility. Because they are stated for all opponents' strategy profiles, they imply their equivalent ex post formulations. In many cases the ex post formulations imply the interim formulations and are easier to verify (this is discussed further in Section 2.3) however since the interim formulations are directly employed in the proof of equilibrium existence, they are given here.

Condition 3A (Uniform upper semicontinuity)**.** *For each agent i there is a function continuous $g^i : \mathbb{R}_+ \times (0,1) \to \mathbb{R}_+$, $g^i(0;\cdot) = 0$, such that for all $s_i \in (0,1)$, all $(\alpha^j)_{j\neq i}$, $a_i \in A^i(s_i)$, and all $\underline{a}_i, \overline{a}_i \in \hat{Y}$ with $\underline{a}_i \leq a_i \leq \overline{a}_i$,*

$$\begin{aligned} U^i\left(\underline{a}_i, \alpha^{-i}; s_i\right) - g^i\left(\left|\left|\underline{a}_i - a_i\right|\right|; s_i\right) &\leq U^i\left(a_i, \alpha^{-i}; s_i\right) \\ &\leq U^i\left(\overline{a}_i, \alpha^{-i}; s_i\right) + g^i\left(\left|\left|\overline{a}_i - a_i\right|\right|; s_i\right). \end{aligned}$$

Condition 3A implies that not only are agent i's upper semicontinuous in her own action, but there is a uniform modulus of semicontinuity that holds acrosss all actions, stategy profiles, and signals. This condition is not in itself obviously satisfied in most auction models, and exposes the role of the type-dependent action spaces A^i. Equilibrium in the model $\mathcal{M}$ will be constructed as an equilibrium in which actions can be taken only from $A^i(s_i)$. In situations where this condition is not binding—where, for example, it can be known exogenously that equilibrium actions lie in a particular set (see Condition 8B)—this is unimportant; in other cases, equilibrium will be appropriately constrained.

While discontinuities are common in auction models, they frequently take a form in which small upward deviations do not yield discretely negative gains (an item is won with strictly greater probability at a marginally higher price), and small downward deviations do not yield discretely positive gains (an item is lost with strictly greater probability). These conditions are not globally suitable in the sense that they are valid only with respect to certain regions of the action space; this is discussed further in Section 3.

[20] Frequently, because it is not of techical importance, I assume that sequences converge; formally, the arguments go through when constrained to convergent subsequences. I note when this assumption is used.

Condition 4A (Local better reply availability)**.** *Let* $\langle(\alpha^{j,t})_{j\neq i}\rangle_{t=1}^{\infty}$ *be a sequence of strategies for agents* $j \neq i$*, converging to the strategy profile* $(\alpha^{j,\star})_{j\neq i}$*. For any* $a_i \in A^i(s_i)$ *and any* $\lambda > 0$*, there is* $a_i' \in A^i(s_i)$ *such that* $||a_i' - a_i|| \leq \lambda$*, and*

$$\lim_{t\nearrow\infty} U^i\left(a_i', \alpha^{-i,t}; s_i\right) > U^i\left(a_i, \alpha^{-i,\star}; s_i\right) - \lambda.$$

Condition 4A is payoff security, restricted to local deviations on the part of agent i. In particular, given a strategy for agent i the limit of a sequence of her opponents' actions, there is a nearby strategy for agent i which yields, in the limit, nearly as much utility. In light of Condition 3A, Condition 4A is frequently straightforward to satisfy by considering small upward deviations, but as this is a condition on limits of opponents' strategies it is a standalone property.

Condition 5A (Surplus splitting)**.** *Let* $(\alpha^{k,\star})_{k=1}^{n}$ *be a feasible strategy profile, and let* $\langle(\alpha^{k,t})_{k=1}^{n}\rangle_{t=1}^{\infty}$ *be a sequence of strategies converging to* $(\alpha^{k,\star})_{k=1}^{n}$*. Suppose that there is an agent* i *and a positive-measure set of signals* S_i *such that for all* $s_i \in S_i$*,*

$$\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) > U^i\left(\alpha^{i,\star}\left(s_i\right), \alpha^{-i,\star}; s_i\right).$$

Then there is an agent j*, a positive-measure set* $S_j \subseteq (0,1)$*, and for each* $s_j \in S_j$ *a positive-measure set* $S_{-j}(s_j) \subseteq (0,1)^{n-1}$ *such that, for all* $s_j \in S_j$ *and* $s_{-j} \in Z_{-j}(s_j)$*,*

$$\lim_{t\nearrow\infty} u_z^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) < u_z^j\left(\alpha^{j,\star}\left(s_j\right), \alpha^{-j,\star}\left(s_{-j}\right); s_j, z\right).$$

Condition 5A is a variant of reciprocal upper semicontinuity; it imposes the additional constraint that the action which generates nearly the utility at the limit be itself relatively close to the limit. In particular, it reflects the fact that auctions have winners and losers, and a discontinuous loss in utility by one agent reflects a gain by one of her opponents when facing her. That her opponents do not necessarily face a discrete increase in interim utility allows for the fact that, globally, they might also lose interim utility, hence Condition 5A is weaker than interim reciprocal upper semicontinuity. Its limiting formulation, given in Condition 5B, considers the availability of an occasionally-profitable sequence of nearby deviations for one of agent i's opponents.

Condition 5B (Surplus splitting (limiting))**.** *Let* $(\alpha^{k,\star})_{k=1}^{n}$ *be a feasible strategy profile, and let* $\langle(\alpha^{k,t})_{k=1}^{n}\rangle_{t=1}^{\infty}$ *be a sequence of strategies converging to* $(\alpha^{k,\star})_{k=1}^{n}$*. Suppose that there is an agent* i *and a positive-measure set* $S_i \subseteq (0,1)$ *such that for all* $s_i \in S_i$*,*

$$\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) > U^i\left(\alpha^{i,\star}\left(s_i\right), \alpha^{-i,\star}; s_i\right).$$

Then there is an agent j*, a positive-measure set* $S_j \subseteq (0,1)$*, for each* $s_j \in S_j$ *a positive-measure set* $S_{-j}(s_j) \subseteq (0,1)^{n-1}$*, and for any* $\lambda > 0$ *a sequence* $\langle\hat{\alpha}^{j,t}\rangle_{t=1}^{\infty}$ *with* $\hat{\alpha}^{j,t}(s_j) \in A^j(s_j) \cap D_\lambda(\alpha^{j,\star}(s_j))$ *such that for all* $s_j \in S_j$ *and* $s_{-j} \in S_{-j}(s_j)$*,*

$$\lim_{t\nearrow\infty} u_z^j\left(\hat{\alpha}^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) > \lim_{t\nearrow\infty} u_z^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right).$$

Lemma 1 (Surplus splitting implies limiting surplus splitting)**.** *Under Conditions 3A and 4A, Condition 5A implies Condition 5B.*

Together, the three conditions on interim utility describe an *auction-like* model. The rationale behind this definition is further explored in Section 3.

Definition 1 (Auction-like models)**.** *A model $\mathcal{M}$ is* auction-like *if it satisfies Conditions 3A, 4A, and 5B.*

To obtain equilibrium existence I will construct a sequence of equilibria in ε-discrete models; the following conditions place some restrictions on equilibrium existence and action structure in the discretized models. For these conditions, let $\langle \varepsilon^t \rangle_{t=1}^{\infty}$ be a monotone decreasing sequence, $\varepsilon^t = (\varepsilon^t_D, \varepsilon^t_R) \gg 0$, such that $\varepsilon^t \to 0$.[21]

Condition 6A (Existence of discrete equilibrium)**.** *There is T such that, for all $t \geq T$, $\mathcal{M}^{\varepsilon^t}$ admits a monotone pure-strategy equilibrium.*

Condition 6A is evidently quite strong, but is in many cases simple to verify; it is necessary for the proof of existence in the continuum-action case since equilibrium is constructed as a limit of equilibria of the discretized models, hence there must be equilibria in the discretized models. Conditions 2, 6B, and 7B give sufficient conditions for the satisfaction of Condition 6A (Lemma 7).

Condition 7A (Dynamic action spaces)**.** *For all agents i and signals s_i, $A^{i,\varepsilon^t}(s_i)$ is a complete lattice, closed with respect to the L^1 norm. Furthermore, there is T such that, for all $t \geq T$, all agents i, and all signals s_i, $A^{i,\varepsilon^t}(s_i)$ is a lattice with respect to the pointwise order on $X_R^{X_D}$. Additionally:*

1. *For all $a_i \in A^i(s_i)$, there exists a monotone decreasing sequence $\langle a^t_i \rangle_{t=1}^{\infty}$ converging to a, such that $a^t_i \in A^{i,\varepsilon^t}(s_i)$ and $a^t_i \geq a_i$ for all t;*

2. *For all sequences $\langle a^t_i \rangle_{t=1}^{\infty}$, $a^t_i \in A^{i,\varepsilon^t}(s_i)$, if $a^t_i \to a^\star_i$ then there is $a_i \in A^{i,\varepsilon^t}(s_i)$ such that $||a_i - a^\star_i|| = 0$.*

Condition 7A appears technical but is actually straightforward. The first point requires that any action in $A^i(s_i)$ can be approximated from above arbirarily closely, as the discretization becomes fine; that approximation is from above is closely related to Condition 3A. The second point requires that actions in the discretized models cannot be too far away from actions in the continuum model; together with the second point, this can be viewed as any discretized action must approximate some action in the base model, and any action in the base model can be approximated in sufficiently-fine discretized models.

[21]Conditions 6A and 7A can be stated as "for $\varepsilon \gg 0$ sufficiently small," but this is less general. All that is necessary is that the conditions are satisfied on a particular path to 0, not all paths; nonetheless, in many contexts there is little practical difference.

The final condition is not necessary for the existence of an equilibrium in which, for all agents, almost all signal realizations are best-responding. However, it can be used to demonstrate that there is an equilibrium in which all signal realizations of all agents are best-responding.

Condition 8A (Constraint technicality)**.** *Let* $\underline{A}^i(s_i) = \cup_{s_i' < s_i} A^i(s_i)$*. For all agents* i*, all signals* s_i*, all* $\lambda > 0$*, and all* $a_i \in A^i(s_i) \setminus \underline{A}^i(s_i)$*, there is* $a_i' \in \underline{A}^i(s_i)$ *such that*

$$U^i\left(a_i', \alpha^{-i}; s_i\right) > U^i\left(a_i, \alpha^{-i}; s_i\right) - \lambda.$$

Condition 8A captures the notion that the type-dependent feasible action space $A^i(s_i)$ is not "very" binding. Condition 8B sharpens this feature in a useful way.

Condition 8B (Constraint super-technicality)**.** *For all* $y \in \hat{Y}$*, all strategy profiles* $(\alpha^j)_{j \neq i}$*, and all* $\lambda > 0$*, there exists* $a_i \in A^i(s_i)$ *such that*

$$U^i\left(a_i, \alpha^{-i}; s_i\right) > U^i\left(y, \alpha^{-i}; s_i\right) - \lambda.$$

Lemma 2 (Constraint super-technicality implies constraint technicality)**.** *Condition 8B implies Condition 8A.*

2.3 Alternate assumptions

The preceding conditions are all employed in establishing the existence of a pure-strategy Bayesian-Nash equilibrium, which necessarily involves optimization of interim utility. In many situations, however, the ex post formulations of the conditions are simpler to verify. Roughly, uniform upper semicontinuity (Condition 3A) allows latitude in whether the interim or ex post versions are employed. With each of the alternate conditions there is a lemma clarifying its relation to its interim counterpart; discussion of these conditions can be found in the preceding subsection, and in Section 3.

Condition 3B (Uniform upper semicontinuity (ex post))**.** *For each agent* i *there is a continuous function* $g^i : \mathbb{R}_+ \times (0,1) \to \mathbb{R}_+$*,* $g^i(0; \cdot) = 0$*, such that for all* $s_i \in (0,1)$*, all* $(a_j)_{j \neq i}$*,* $a_i \in A^i(s_i)$*, and all* $\underline{a}_i, \overline{a}_i \in \hat{Y}$ *with* $\underline{a}_i \leq a_i \leq \overline{a}_i$*,*

$$\begin{aligned} u_z^i\left(\underline{a}_i, a_{-i}; s_i\right) - g^i\left(\left|\left|\underline{a}_i - a_i\right|\right|; s_i\right) &\leq u_z^i\left(a_i, a_{-i}; s_i\right) \\ &\leq u_z^i\left(\overline{a}_i, a_{-i}; s_i\right) + g^i\left(\left|\left|\overline{a}_i - a_i\right|\right|; s_i\right). \end{aligned}$$

Lemma 3 (Equivalence of uniform upper semicontinuity)**.** *Condition 3A is satisfied if and only if Condition 3B is satisfied.*

Condition 4B (Local better reply availability (ex post))**.** *Let* $\langle (a_{j,t})_{j \neq i} \rangle_{t=1}^{\infty}$ *be a sequence of actions for agents* $j \neq i$*, converging to* $(a_{j,\star})_{j \neq i}$*. For any* $a_i \in A^i(s_i)$ *and any* $\lambda > 0$*, there is* $a_i' \in A^i(s_i)$ *such that* $||a_i' - a_i|| \leq \lambda$*, and*

$$\lim_{t \nearrow \infty} u_z^i\left(a_i', a_{-i,t}; s_i\right) > u_z^i\left(a_i, a_{-i,\star}; s_i\right) - \lambda.$$

Lemma 4 (Equivalence of local better reply availability)**.** *Under Condition 3B, Condition 4B is satisfied if and only if Condition 4A is satisfied.*

Condition 5C (Surplus splitting (ex post))**.** *Let $(\alpha^{k,\star})_{k=1}^{n}$ be a feasible strategy profile, and let $\langle(\alpha^{k,t})_{k=1}^{n}\rangle_{t=1}^{\infty}$ be a sequence of strategies converging to $(\alpha^{k,\star})_{j=1}^{n}$. Suppose that there is an agent i and a positive-measure set of signals $S_i \subseteq (0,1)$ and a positive-measure set $S_{-i}(s_i) \subseteq (0,1)^{n-1}$ for each $s_i \in S_i$ such that for all $s_i \in S_i$ and all $s_{-i} \in S_{-i}(s_i)$,*

$$\lim_{t\nearrow\infty} u_z^i\left(\alpha^{i,t}(s_i), \alpha^{-i,t}(s_{-i}); s_i\right) > u_z^i\left(\alpha^{i,\star}(s_i), \alpha^{-i,\star}(s_{-i}); s_i\right).$$

Then there is an agent j and a positive-measure set $S_j \subseteq (0,1)$, and for each $s_j \in S_j$ a positive-measure set $S_{-j}(s_j) \subseteq (0,1)^{n-1}$ such that, for all $s_j \in S_j$ and all $s_{-j} \in S_{-j}(s_j)$,

$$\lim_{t\nearrow\infty} u_z^j\left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j\right) < u_z^j\left(\alpha^{j,\star}(s_j), \alpha^{-j,\star}(s_{-j}); s_j\right).$$

Lemma 5 establishes that Condition 5C implies Condition 5A. That this implication is not bidirectional follows from the fact that agent i's ex post utility can be increasing against a particular set of opponents while it is globally decreasing. Bidirectional implication can be reinstated if the feasibility constraint is ignored, by simulating the conditional strategies played by some subset of $-i$'s signal realizations, but this is outside the model. If there exists for each agent some nondistortionary outside option, bidirectional implication can be shown by analyzing these conditional strategies.

Lemma 5 (Ex post surplus splitting implies interim surplus splitting)**.** *Condition 5C implies Condition 5A.*

Condition 5D (Surplus splitting (limiting ex post))**.** *Let $(\alpha^{k,\star})_{k=1}^{n}$ be a feasible strategy profile, and let $\langle(\alpha^{k,t})_{k=1}^{n}\rangle_{t=1}^{\infty}$ be a sequence of strategies converging to $(\alpha^{k,\star})_{k=1}^{n}$. Suppose that there is an agent i and a positive-measure set $S_i \subseteq (0,1)$, and for each $s_i \in S_i$ a positive-measure set $S_{-i}(s_i) \subseteq (0,1)^{n-1}$ such that for all $s_i \in S_i$ and $s_{-i} \in S_{-i}(s_i)$,*

$$\lim_{t\nearrow\infty} u_z^i\left(\alpha^{i,t}(s_i), \alpha^{-i,t}(s_{-i}); s_i\right) > u_z^i\left(\alpha^{i,\star}(s_i), \alpha^{-i,\star}(s_{-i}); s_i\right).$$

Then there is an agent j, a positive-measure set $S_j \subseteq (0,1)$, for each $s_j \in S_j$ a positive-measure set $S_{-j}(s_j) \subseteq (0,1)^{n-1}$, and for any $\lambda > 0$ a sequence $\langle\hat{\alpha}^{j,t}\rangle_{t=1}^{\infty}$ with $\hat{\alpha}^{j,t}(s_j) \in A^j(s_j) \cap D_\lambda(\alpha^{j,\star}(s_j))$ such that for all $s_j \in S_j$ and $s_{-j} \in S_{-j}(s_j)$,

$$\lim_{t\nearrow\infty} u_z^j\left(\hat{\alpha}^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j\right) > \lim_{t\nearrow\infty} u_z^j\left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j\right).$$

Lemma 6 (Limiting ex post surplus splitting implies limiting interim surplus splitting)**.** *Condition 5D implies Condition 5B.*

The conditions on the structure of the discretized action spaces can be strengthened so that they are easier to verify.

Condition 6B (Single crossing and quasisupermodularity)**.** *For each agent i and all signals s_i, u^i satisfies single crossing in agent i's own action and signal, and quasisupermodularity in agent i's own action,*

$$\begin{aligned}
&U^i\left(y_i, \alpha^{-i}; s_i\right) \geq (>)\, U^i\left(y_i \wedge y_i', \alpha^{-i}; s_i\right) \\
&\qquad \implies U^i\left(y_i \vee y_i', a_{-i}; s_i\right) \geq (>)\, U^i\left(y_i', \alpha^{-i}; s_i\right), \text{ and} \\
s_i' > s_i,\ y' > y,\ &U^i\left(y_i', \alpha^{-i}; s_i\right) \geq (>)\, U^i\left(y_i, \alpha^{-i}; s_i\right) \\
&\qquad \implies U^i\left(y_i', \alpha^{-i}; s_i'\right) \geq (>)\, U^i\left(y_i, \alpha^{-i}; s_i'\right).
\end{aligned}$$

Condition 7B (Structure of action space)**.** *For each agent i and all signals $s_i \in (0,1)$, $A^i(s_i)$ is a complete lattice with respect to the pointwise order on $X_R^{X_D}$, and all sequences in $A^i(s_i)$ which converge (with respect to the L^1 norm on X_D) in $\hat{Y}$ also converge to some $a^\star \in A^i(s_i)$.*

A lattice structure on i's feasible action space ensures that the supremum of two feasible actions will always be feasible. Since Condition 3A stipulates that upward deviations cannot be disproportionately harmful, the availability of upward deviations is key to the analysis of the model.

Condition 7B interacts closely with Condition 3A, 4A, and 5A; because of this, the requirement that $A^i(s_i)$ be a lattice is not necessarily trivial. If actions must fall weakly below some bound, the supremum of any two actions is also below this bound and hence the feasible action set is a lattice; if instead the integral of actions must fall weakly below some bound, the supremum of two actions may be above this bound and hence the feasible action set is not an upper semilattice, and hence not a lattice. How to relax this structural requirement is an interesting avenue for future research.

Condition 9 (Availability of discrete approximations)**.** *There is an $\delta > 0$ such that, for all agents i, all signals s_i, and all $\delta' \in (0, \delta)$, $A^{i,(\delta',\delta')}(s_i) \subseteq \hat{Y}$.*

Condition 9 is strictly technical, and is simple to satisfy: in any model, the set $\hat{Y}$ can be easily enlarged by expanding X_R. This has no effect on incentives, outside of needing to ensure that utilities are well-defined on the larger set.

Lemma 7 (Implications of action space assumptions)**.** *Let $\langle \varepsilon^t \rangle_{t=1}^{\infty}$ be a strictly decreasing sequence, $\varepsilon^t \searrow (0,0)$. Then Conditions 6B, 7B, and 9 imply Conditions 6A and 7A with respect to the ε^t-discrete models $\langle \mathcal{M}^{\varepsilon^t} \rangle_{t=1}^{\infty}$.*

3 Example: first-price auctions for a single unit

To explore the role of each of the above conditions, consider a 2-bidder first-price auction for a single unit, where each bidder i has a private value s_i independently distributed uniformly on $(0,1)$. This auction admits a unique pure-strategy equilibrium in which each agent bids half her value. Exogenous randomness in this model takes the form of tiebreaking when necessary; in this case, Supp $Z = \{(1,0),(0,1)\}$, and $\Pr(z = \hat{z} \in Z) = 1/2$. If bids are (b_1, b_2), the quantity allocation is

$$q(b_1, b_2; z) = \begin{cases} (1,0) & \text{if } b_1 > b_2, \\ (0,1) & \text{if } b_2 > b_1, \\ z & \text{otherwise.} \end{cases}$$

There are two natural ways of mapping monotone functions into single-unit bids. In the first case, a bid is a constant function on its domain, and the "bid" is the constant value of the function. In the second case, a bid is a degenerate cumulative distribution function over random bids.

3.1 Singleton domain

Proofs are found in Appendix C.1.

Let $X_D = \{0\}$, and $X_R \subset \mathbb{R}$.[22] Bidder i's ex post utility is

$$u_z^i\left(b^i, b^{-i}; s_i\right) = \left(s_i - b^i(0)\right)\left(1\left[b^i > b^{-i}\right] + \frac{1}{2}1\left[b^i = b^{-i}\right]\right).$$

Denote this auction model by $\mathcal{M}^0$.

Condition 1 (Utility and actions)**.** Utility is increasing and continuous in signal, and is bounded above by $\max\{s_i - \min X_R, 0\}$ and below by $\min\{-\max X_R, 0\}$. Since actions $y \in \hat{Y}$ can be uniquely identified with real numbers $x \in \mathbb{R}$, the action space is a complete lattice as long as it is closed.

Condition 2 (Imitability)**.** This is easily satisfied if, among other possible properties, there is $R^i(s_i) = [\underline{r}^i(s_i), \overline{r}^i(s_i)]$, $\underline{r}^i$ monotone decreasing and $\overline{r}^i$ monotone increasing, such that

$$A^i(s_i) = \left\{y \in \hat{Y} : y(0) \in R^i(s_i)\right\}.$$

One natural restriction, which will be employed in the discussion of Condition 3B, is that $A^i(s_i)$ is the set of all "bids" which are both weakly positive and weakly below the agent's value for the item. In a symmetric first-price auction this restriction is without loss.

Condition 3B (Uniform upper semicontinuity (ex post))**.** To meet the above conditions, let $X_R = [0,2]$ and $A^i(s_i) \subseteq [0,1]$; that is, it is never rationalizable for an agent to outbid her opponent's highest value, so actions can be taken from the closure of the set of her opponent's values.

[22] It is straightforward to consider nondegenerate domains. In particular, given an arbitrary domain X_D and a monotone function $y \in \hat{Y}$, the interpreted "bid" in the model could be $\sup_{x \in X_D} y(x)$, or $\int_{X_D} y(x)dx$. The issue of interpretation in this context is not of economic importance and is ignored.

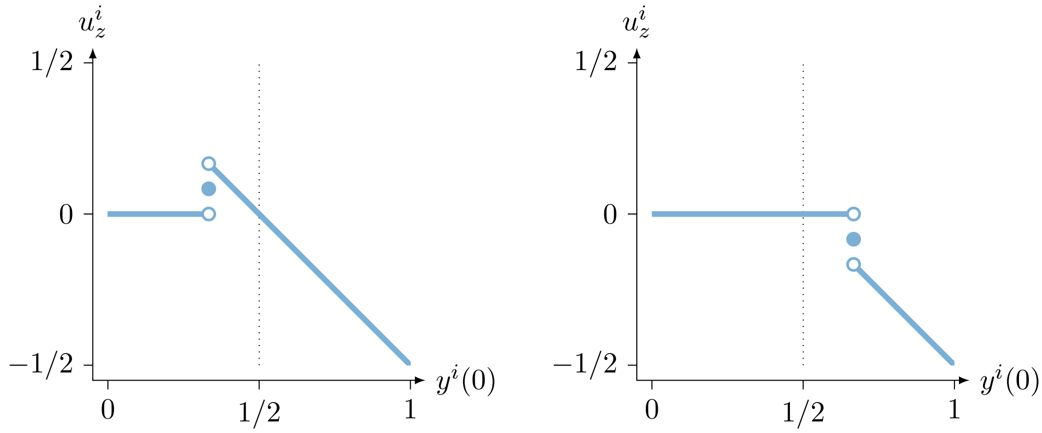

Figure 1: Bidder i's ex post utility is upper semicontinuous in her own bid only when the feasible action space is appropriately constrained. In both panels, $s_i = 1/2$; the left panel is agent i's utility when facing $b^j = 1/3$, and the right panel is agent i's utility when facing $b^j = 2/3$. A discontinuous downward jump in utility from a small increase in bid cannot occur when $b^i \leq s_i$.

Let $s_i \in (0, 1)$, and suppose that $y \in A^i(s_i)$ is such that $y(0) \in (s_i, 1]$. Then for any $\lambda > 0$,

$$\begin{aligned} u_z^i\left(y, y; s_i\right) &= \frac{1}{2}\left(s_i - y\left(0\right)\right) \\ &> \left(s_i - \left[y\left(0\right) + \lambda\right]\right) = u_z^i\left(y + \lambda, y; s_i\right). \end{aligned}$$

In particular, $\lim_{\lambda \searrow 0} u_z^i(y + \lambda, y; s_i) - u_z^i(y, y; s_i) = (s_i - y(0))/2 < 0$, contradicting Condition 3B.

This contradiction relies on the ability to select y such that $y(0) > s_i$. This is easily addressed by refining A^i,

$$A^i\left(s_i\right) = \left\{y \in \hat{Y} : \ y\left(0\right) \leq s_i\right\}.$$

Lemma 23 formalizes this result, which is intuitive: since payments (conditional on winning) are continuous in bid, a small increase in agent i's bid can discretely affect her utility only if it discretely affects her allocation. Since she is constrained to bid below her value, a discontinuous change in her allocation is always utility-positive. Similarly, a slight decrease in bid can only discretely affect agent i's utility if it causes her to lose the item, which is always utility-negative.

In many auction contexts a restriction of bids so that they are below values is feasible and without loss (with regard to equilibrium existence). This will be explored further in Section 5.

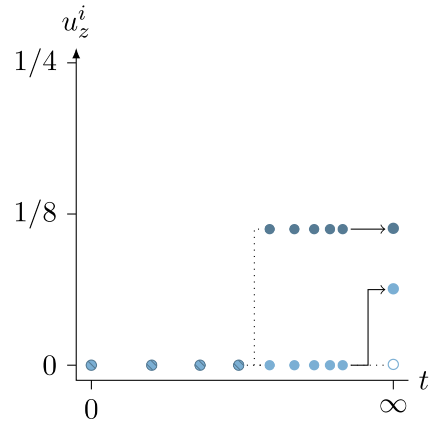

Figure 2: If bidder i's utility jumps discretely at a convergent limit of her opponent's strategies, she can increase her bid slightly to achieve almost as much utility in the limit as at the limit of the original sequence. In this Figure, $s_i = 1/2$ and $b^{j,t} = 1/4 + 1/4^t$. Light blue dots are bidder i's utility from bidding $b^i = 1/4$, and dark blue dots are bidder i's utility from bidding $\hat{b}^i = 1/4 + \varepsilon$, for small $\varepsilon > 0$.

Condition 4B (Local better reply availability (ex post))**.** Because actions here are uniquely identified with real numbers, Condition 4B follows more or less immediately from Condition 3B. If, given a sequence of her opponent's actions, agent i's utility converges or drops discontinuously, the condition is automatically satisfied (letting the alternate action a_i' equal the original action a_i). Otherwise, her utility jumps upward at the limit.

In a first-price auction with bids below values, bidder i's utility can jump upward at the limit only if i's allocation probability jumps discontinuously upward at the limit; in turn, this implies that there is tiebreaking at the limit. At this point agent i can guarantee herself a certain allocation with a small increase in her bid, associated with a proportionately small increase in payment. Then near the limit of her opponent's bids she is winning the item with certainty and paying only slightly more, hence in the limit she attains nearly the utility of the original action at the limit; see Figure 2. This is formalized in Lemma 24.

Condition 5C (Surplus splitting (ex post))**.** As with Condition 4B, when bids are bounded above by agents' values the only way for an agent to lose utility at a limit is if her opponent, at the limit, is submitting a bid equal to hers, while near the limit her opponent's bids were strictly lower than her own. By market clearing, this implies that her opponent is gaining some probabilistic share of the item, and since values are strictly increasing in signal it follows that some subset of her opponent's valuations witness a discrete utility increase at the limit. This is formally verified in Lemma 25; some care must be taken to ensure that an increase in allocation probability is associated with an increase in ex post utility and not, for example,

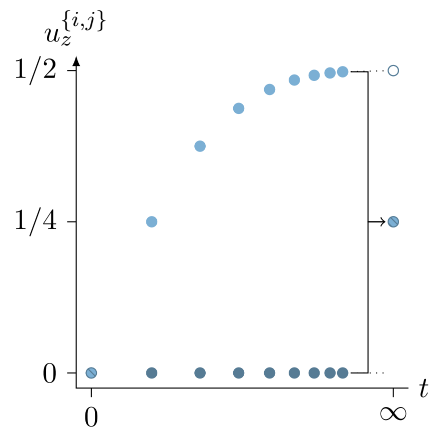

Figure 3: If bidder i's utility falls discretely at a convergent limit of actions, bidder j's utility jumps discretely at the same limit due to tiebreaking.[24] In this Figure, $s_i, s_j = 1/2$; $b^{i,t} = 1/2^t$ and $b^{j,t} = 0$. Light blue dots are bidder i's utility and light blue squares are bidder i's bid; and dark blue dots are bidder i's utility and dark blue squares are bidder j's bid.

bidding away all of one's potential surplus.

Condition 6B (Single crossing and quasisupermodularity)**.** Ex post utility is continuous and weakly increasing in signal. Satisfaction of single-crossing and quasisupermodularity is discussed in McAdams (2003), among many others.

Condition 7B (Structure of action space)**.** This is easily satisfied as long as each $A^i(s_i)$ is a closed subset of $\hat{Y}$; that is, there exists a closed subset of $R \subseteq \mathbb{R}$ such that

$$A^i(s_i) = \left\{ y \in \hat{Y} : y(0) \in R \right\}.$$

Then the relatively mild construction suggested to satisfy Condition 2 will also satisfy Condition 7B. Note that this is not a restriction on the feasible action spaces themselves, but rather on the set from which they are formed; in particular, utility can be reasonably evaluated on a set that is at least somewhat larger than the feasible action spaces.

Condition 8A (Constraint technicality)**.** Note that $\underline{A}^i(s_i) = \{y \in \hat{Y} : y(0) \in [0, s_i)\}$, and hence $A^i(s_i) \setminus \underline{A}^i(s_i) = \{y \in \hat{Y} : y(0) = s_i\}$. Then $u_z^i(y, y^{-i}; s_i) = 0$ for all $y \in A^i(s_i) \setminus \underline{A}^i(s_i)$, and $u_z^i(y', y^{-i}; s_i) \geq u_z^i(y, y^{-i}; s_i)$ for all $y' \in \underline{A}^i(s_i) \cap D_\lambda(y)$.[25] That is, the unique element of the frontier of agent i's type-dependent feasible action space represents a bid that is equal to her value for the item, which yields zero utility whether she wins or loses the item; actions strictly within her strategy space yield

[24] Condition 5C is given in terms of strategies, but for visualization's sake this plot is in terms of actions.

[25] $\underline{A}^i(s_i) \neq \emptyset$ since $s_i \in (0, 1)$.

at least this much utility, by construction. This inequality is then trivially satisfied for any strategy α^{-i}.

As with Condition 3B, satisfaction of Condition 8A is relatively straightforward when the feasible action sets are judiciously selected.

Condition 9 (Availability of discrete approximations)**.** Consider type-dependent action spaces given by

$$A^i(s_i) = \left\{ y \in \hat{Y} : y(0) \in \left[0, \overline{r}^i(s_i)\right] \right\}.$$

Suppose that $\overline{r}^i$ is bounded, so that there is $r^\star > \max_j \sup_{s_j} \overline{r}^j(s_j)$; let $X_R = [0, r^\star]$. So long as $\varepsilon < r^\star - \max_j \sup_{s_j} \overline{r}^j(s_j)$, $A^{j,\varepsilon}(s_j) \subset \hat{Y}$. Note that if instead $r^\star = \max_i \sup_{s_j} \overline{r}^j(s_j)$, for any $\varepsilon > 0$ there is an agent i, a signal s_i, and an $\varepsilon' \in (0, \varepsilon)$ such that $\lceil r^i(s_i)/\varepsilon' \rceil \varepsilon' > r^\star$, hence this construction will not satisfy Condition 9. Given that values are between 0 and 1, one natural restriction is that bids must be between 0 and 2, $X_R = [0, 2]$. This addresses the fact that the space in which the feasible action sets are embedded can be determined essentially arbitrarily; frequently, then, satisfaction of this condition is a formality.

3.2 Probabilistic approach

Proofs are found in Appendix C.2.

Alternatively, the action space can be conceptualized as a set of distributions over bids, or of mixed strategies. In particular, let $X_D = [0, 1]$ and $X_R = [0, 2]$, and for each agent i and signal s_i let the feasible action space be

$$A^i(s_i) = \left\{ y \in \hat{Y} : \ y \text{ decreasing}, y(s_i) = 1, y(s_i) = 0 \right\}.$$

Note that although CDFs are monotone increasing functions, actions here are monotone decreasing functions; this difference will be accounted for in utility calculations.[26]

An action $a_i \in A^i(s_i)$ may be translated to a CDF by

$$[\tau(a_i)](x) = \begin{cases} 1 - a_i(x) & \text{if } a_i \text{ is continuous at } x, \\ \inf_{x' > x} 1 - a_i(x') & \text{otherwise.} \end{cases}$$

That is, $\tau(a_i)$ is the right-continuous function closest to and above $1 - a_i$. Under this translation, it will be sufficient to analyze the model in terms of standard probability

[26]This gets at an important issue in the proof of existence: uniform upper semicontinuity is a tool for knowing how, exactly, an agent might find profitable deviations near the limit. It would do just as well to assume uniform lower semicontinuity, or even uniform semicontinuity on an idiosyncratic, agent-by-agent basis. These cases can be handled simply but make the proof mechanism more complicated than necessary. With these additional results, bids could be represented as proper CDFs; without these results, representing actions as monotone increasing functions means that a small increase in action corresponds to a bid distribution which is stochastically dominated, which can (for example, in the case of mass points) yield a discontinuous drop in utility.

distributions: if $y, y' \in \hat{Y}$, then $||y - y'|| = ||\tau(y) - \tau(y')||$, and the inequalities in Condition 3B must be reversed, since $y \geq y'$ implies $\tau(y) \leq \tau(y')$. Similarly, single-crossing becomes negative single-crossing, and quasisupermodularity becomes quasisubmodularity. For the remainder of this subsection I will treat actions as CDFs and ignore the translation function.[27]

In this case, bidder i's "ex post" utility is

$$u_z^i\left(a_i, a_{-i}; s_i\right) = \int_{X_D} \int_{X_D} \left(s_i - x_i\right) q^i\left(x_i, x_{-i}\right) da_{-i}\left(x_{-i}\right) da_i\left(x_i\right).$$

This can be represented in a straightforward way by letting model randomness contain two independent random variables for selection of bids from the distributions (a_i, a_{-i}). The quantity she is allocated, conditional on realized bids (x_i, x_{-i}) is

$$q^i\left(x_i, x_{-i}\right) = \begin{cases} 1 & \text{if } x_i > x_{-i}, \\ \frac{1}{2} & \text{if } x_i = x_{-i}, \\ 0 & \text{if } x_i < x_{-i}. \end{cases}$$

Denote this auction model by $\mathcal{M}^\sigma$.

The basic conditions on the structure of the action space are satisfied as before. Once the remaining conditions are verified, Theorem 1 implies that this model admits a pure-strategy Bayesian-Nash equilibrium. Since actions in this model are equivalent to mixed strategies, the implication is that first-price auctions admit mixed-strategy Bayesian-Nash equilibria, which is a direct implication of the (known) existence of pure-strategy equilibria.

Condition 3B (Uniform upper semicontinuity (ex post))**.** As mentioned, since mixed strategies are negative affine transformations of the actions in the model, it is necessary to verify uniform *lower* semicontinuity. Roughly, a small change in CDF to a stochastically-dominant distribution will always positively affect allocation probabilities; because the shift is small, it cannot have an outsized effect on expected payment (conditional on winning), hence utility is either discretely increased or varies continuously in this rightward shift. Discontinuities will only arise when the CDFs implicit in actions contain mass points which are aligned between the two agents. This is formalized in Lemma 26.

Condition 4B (Local better reply availability (ex post))**.** As in the case with a singleton domain, if agent i's utility converges or discontinuously falls at a limit of her opponent's strategies, Condition 4B is trivially satisfied. If her utility jumps upward, it must be that she is now winning ties against her opponent with some positive probability; that is, there is at least one shared mass point x in each of their CDFs $a_i, a_{-i,\star}$. If this is the case, a small rightward shift in her distribution is sufficient to win this tie while only slightly increasing payment, hence she is able

[27] With additional work, it can be shown that the translation is robust to the fact that an arbitrary monotone function in $\hat{Y}$ must be renormalized to describe a distribution. I elide this detail here.

to find a small deviation which locks in most of the utility she obtained at the limit when employing her original action. This is formalized in Lemma 27.

Condition 5C (Surplus splitting (ex post))**.** As with Condition 4B, agent i's utility can fall discretely at a limit of her and her opponent's strategies only if tiebreaking arises at the limit. This implies that, at the limit, her opponent must be receiving a discretely greater quantity allocation; if this happens to agent i with positive probability, then there is a positive mass of her opponent's types that witness an increase in quantity at the limit. Properties of monotone functions ensure that for most of these type realizations for her opponent, the mass point is strictly below her value for the object, and hence the increased allocation is utility-positive. This is shown in Lemma 28.

Condition 6B (Single crossing and quasisupermodularity)**.** Satisfaction of single-crossing and quasisupermodularity follows for essentially the same reasons as in other auction contexts, and is proved formally in Lemma 29.

It is worth noting that the verification of these conditions did not hinge on properties of the CDF other than monotonicity; then if CDFs were restricted to point masses—for each $a_i \in A^i(s_i)$, there is c such that $a_i(x) = 1[x \geq c]$—all results would hold, and instead of suggesting the existence of mixed-strategy equilibria (pursuant to Theorem 1; here, a pure-strategy CDF is a mixed strategy) this would imply the existence of a pure-strategy equilibrium in a game without probability distributions.

4 Results: equilibrium existence and approximation

Proofs are found in Appendix B.

The construction providing equilibrium existence and approximation begins by looking at a sequence of equilibria in ε-discretized models. For $\varepsilon \gg 0$ sufficiently small, the ε-discrete model $\mathcal{M}^\varepsilon$ either admits a monotone pure-strategy equilibrium by assumption (Condition 6A) or satisfies the conditions for equilibrium existence set forth in Reny (2011) (Conditions 6B, 7B, and 9, together with finiteness of $A^{i,\varepsilon}$).

In what follows, let $\langle \varepsilon^t \rangle_{t=1}^\infty$ be a sequence in $\mathbb{R}^2_{++}$ converging to $(0, 0)$.

Lemma 8 (Pure-strategy equilibrium in $\mathcal{M}^\varepsilon$)**.** *Suppose that the ε^t-discretizations of $\mathcal{M}$ satisfy Condition 2 and either Condition 6A, or Conditions 6B, 7B, and 9. Then there is T such that for all $t > T$ there is a monotone pure-strategy Bayesian-Nash equilibrium $(\alpha^{i,t})_{i=1}^n$ of the model $\mathcal{M}^{\varepsilon^t}$, in which for each agent i and almost all signals s_i,*

$$U^i\left(\alpha^i\left(s_i\right), \alpha^{-i}; s_i\right) \geq \max_{a_i \in A^{i,\varepsilon^t}(s_i)} U^i\left(a_i, \alpha^{-i}; s_i\right).$$

As the cornerstone on which the remaining results on built, I will henceforth assume that the antecedents of Lemma 8 are satisfied. For any t, let $(\alpha^{i,t})$ be a pure-strategy Bayesian-Nash equilibrium in $\mathcal{M}^{\varepsilon_t}$. Since each $\alpha^{i,t}$ is bounded, selection results (c.f. Widder (1941)) imply that there is a pointwise limit on any countable set of points.[28] It is useful that this set of points be dense, hence let $\mathcal{X}_D = X_D \cap \mathbb{Q}$ and $\mathcal{S} = (0,1) \cap \mathbb{Q}$.

Lemma 9 (Pointwise convergence)**.**[29] *There is a strategy profile* $(\alpha^{i,\square})$ *such that for all agents* i*, all* $x \in \mathcal{X}_D$*, and all* $s \in \mathcal{S}$*,*

$$\lim_{t \nearrow \infty} \left[\alpha^{i,t}(s)\right](x) = \left[\alpha^{i,\square}(s)\right](x).$$

The sets $\mathcal{X}_D$ and $\mathcal{S}$ are countable, hence the strategy profile $(\alpha^{i,\square})_{i=1}^n$ comprised of functions on X_D may have significant "holes." However, monotone functions on compact domains are continuous almost everywhere (Lavrič, 1993), meaning that any monotone function that coincides with $\alpha^{i,\square}$ where it is uniquely defined will be L^1-equivalent to $\alpha^{i,\square}$.

Lemma 10 (Convergence to limit)**.**[30] *For all agents* i*,* $||\hat{\alpha}^i - \alpha^{i,\square}|| = 0$ *implies*

$$\lim_{t \nearrow \infty} \left|\left|\alpha^{i,t} - \hat{\alpha}^i\right|\right| = 0.$$

Furthermore, for almost all $s_i \in (0,1)$*,*

$$\lim_{t \nearrow \infty} \left|\left|\alpha^{i,t}(s_i) - \hat{\alpha}^i(s_i)\right|\right| = 0.$$

This permits the construction of *supremum-limit* strategies $\overline{\alpha}^i$.[31] These are strategies at which each type realization s_i is playing an action that is the least upper bound of actions for all lower type realizations $s_i' < s_i$.

Definition 2 (Supremum strategy)**.** α^i *is a* supremum *strategy for agent* i *if for all* $s_i \in (0,1)$*,*

$$\alpha^i(s_i) = \sup_{s_i' < s_i} \alpha^i\left(s_i'\right).$$

Definition 3 (Supremum-limit strategy)**.** $\overline{\alpha}^i$ *is a* supremum-limit *strategy for agent* i *if for all* $s_i \in (0,1)$*,*

$$\overline{\alpha}^i(s_i) = \sup_{s_i' < s_i} \alpha^{i,\square}\left(s_i'\right).$$

As strategies converge, so too does utility for almost all agents. This is the bulk of the proof of equilibrium existence.

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[28] In a single dimension, Helly's selection theorem guarantees that any sequence of bounded monotone functions on a compact domain admits a convergent subsequence. In multiple dimensions these results appeal to total boundedness, which is not exogenously guaranteed in auction-like models; instead, pointwise convergence and monotonicity are used to derive L^1 convergence.

[29] In Appendix B this is proved as Lemma 14.

[30] In Appendix B this is proved as Lemmas 15 and 16.

[31] The term "limit-supremum strategy" is more grammatically apt, but is not used to distinguish these strategies from the mathematical notion of limit supremum.

Lemma 11 (Utility convergence almost everywhere).[32] *For each agent i and almost all signals $s_i \in (0,1)$ such that $||\alpha^{i,t}(s_i) - \overline{\alpha}^i(s_i)|| \to 0$,*

$$\lim_{t \nearrow \infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) = U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right).$$

Since Lemma 11 holds for almost all s_i with $\alpha^{i,t}(s_i) \to \overline{\alpha}^i(s_i)$, and Lemma 10 says that almost all s_i satisfy this convergence property, for all agents almost all signal realizations have convergent utility. One side of Lemma 11 is a direct consequence of Conditions 3A and 4A: if the value in the limit is below the value at the limit, a slight upward deviation will yield discretely more utility in some discretized $\mathcal{M}^{\varepsilon^t}$. The other side follows from Condition 5A: if agent i's utility falls at the limit, one of her opponents has an available deviation which will yield discretely greater utility in some discretized $\mathcal{M}^{\varepsilon^t}$.

Returning to Condition 4A gives mutual best-responsiveness.

Theorem 1 (Equilibrium existence (almost everywhere)). *The supremum-limit strategy profile $(\overline{\alpha}^i)_{i=1}^n$ forms a monotone pure-strategy Bayesian-Nash equilibrium in which for each agent i and almost all signal realizations i,*

$$U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) \geq \sup_{a_i \in A^i(s_i)} U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right).$$

Corollary 1 (Symmetric equilibrium). *If $A^i = \hat{A}$ and $u^i = \hat{u}$ for all agents i, then $\mathcal{M}$ admits a symmetric monotone pure-strategy Bayesian-Nash equilibrium $(\hat{\alpha})_{i=1}^n$.*

In particular, if agent i has an action a_i which discretely improves on $\overline{\alpha}^i(s_i)$, she has an action close to a_i which is an improvement over some $\alpha^{i,t}$ against $(\alpha^{j,t})_{j\neq i}$. For ε^t sufficiently small, a_i can be approximated into $A^{i,\varepsilon_t}(s_i)$ at a loss that is of order $g^i(\max_x \varepsilon_x^t; s_i)$ (Condition 3A). Since $(\alpha^{j,t})_{j=1}^n$ is a Bayesian-Nash equilibrium in which almost all signal realizations are best-responding, this is a contradiction if a positive mass of agents have utility-improving actions.

Condition 8A permits the equilibrium construction to apply to all agents, demonstrated in Lemma 22.

Theorem 2 (Equilibrium existence (everywhere)). *Let $(\overline{\alpha}^i)_{i=1}^n$ be a monotone pure-strategy Bayesian-Nash equilibrium in supremum strategies such that for almost all $s_i \in (0,1)$,*

$$U^i\left(\alpha^i\left(s_i\right), \alpha^{-i}; s_i\right) \geq \sup_{a_i \in A^i(s_i)} U^i\left(a_i, \alpha^{-i}; s_i\right).$$

Then if Condition 8A is satisfied, the strategy profile $(\alpha^i)_{i=1}^n$ constitutes a pure-strategy Bayesian-Nash equilibrium in which for each agent i and all signal realizations $s_i \in (0,1)$,

$$U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) \geq \sup_{a_i \in A^i(s_i)} U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right).$$

[32] In Appendix B this is proved as Lemmas 18 and 19.

Corollary 2 (Equilibrium existence in supremum-limit strategies)**.** *If Condition 8A is satisfied and* $(\overline{\alpha}^i)_{i=1}^n$ *is a supremum-limit strategy profile, then* $(\alpha^i)_{i=1}^n$ *constitutes a monotone pure-strategy Bayesian-Nash equilibrium in which for each agent* i *and all signal realizations* $s_i \in (0,1)$*,*

$$U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) \geq \sup_{a_i \in A^i(s_i)} U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right).$$

If instead of Condition 8A, Condition 8B is satisfied[33] equilibrium existence can be extended to all signals, and strategies remain best responses even when the type-dependent action spaces are relaxed to homogenous action spaces, $A^i(s_i) = \hat{Y}$.

Theorem 3 (Unconstrained equilibrium existence)**.** *Let* $(\overline{\alpha}^i)_{i=1}^n$ *be a monotone pure-strategy Bayesian-Nash equilibrium of the model* $\mathcal{M} = (n, u, X, A, F, Z)$ *such that for all* $s_i \in (0,1)$*,* $\alpha^i(s_i) = \sup_{s_i' < s_i} \alpha^i(s_i')$*, and for almost all* $s_i \in (0,1)$*,*

$$U^i\left(\alpha^i\left(s_i\right), \alpha^{-i}; s_i\right) \geq \sup_{a_i \in A^i(s_i)} U^i\left(a_i, \alpha^{-i}; s_i\right).$$

Then if Condition 8B is satisfied, the strategy profile $(\alpha^i)_{i=1}^n$ *constitutes a pure-strategy Bayesian-Nash equilibrium in the model* $\mathcal{M}' = (n, u, X, \hat{A}, F, Z)$*, where* $\hat{A}^i(s_i) = \hat{Y}$*, such that for each agent* i *and all signal realizations* $s_i \in (0,1)$*,*

$$U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) \geq \sup_{a_i \in \hat{Y}} U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right).$$

Corollary 3 (Unconstrained equilibrium existence in supremum-limit strategies)**.** *If Condition 8B is satisfied and* $(\overline{\alpha}^i)_{i=1}^n$ *is a supremum-limit strategy profile, then* $(\alpha^i)_{i=1}^n$ *constitutes a monotone pure-strategy Bayesian-Nash equilibrium in which for each agent* i *and all signal realizations* $s_i \in (0,1)$*,*

$$U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) \geq \sup_{a_i \in \hat{Y}} U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right).$$

4.1 Equilibrium approximation

The construction of equilibrium in $\mathcal{M}$ as a profile supremum-limit strategies suggests that equilibrium in $\mathcal{M}$ may be near equilibrium in the ε^t-discretization $\mathcal{M}^{\varepsilon^t}$.[34] In these results, I assume that $\langle(\alpha^{i,t})_{i=1}^n\rangle_{t=1}^\infty$ is a sequence of monotone pure-strategy Bayesian-Nash equilibria of the ε^t-discrete models $\mathcal{M}^{\varepsilon^t}$ converging to the supremum-limit strategy profile $(\overline{\alpha}^i)_{i=1}^n$.

[33] Lemma 2 establishes that Condition 8B implies Condition 8A.

[34] In models in which equilibrum is unique this approximation is strict, in the sense that all equilibria converge to the unique equilibrium in $\mathcal{M}$; however without further results on uniqueness the strongest statement possible is, "The sequence of equilibria in $\mathcal{M}^{\varepsilon^t}$ contains a convergent subsequence which converges to an equilibrium of $\mathcal{M}$."

Definition 4 (Utility-relevant function)**.** *Let (W, T_W) be a topological space, and let $\langle y^t \rangle_{t=1}^{\infty}$ be a sequence of strategy profiles converging to $y^\star$. A function $o : \hat{Y}^n \times \mathit{Supp}\ Z \to W$ is* utility-relevant *if, for all $z \in \mathit{Supp}\ Z$, whenever there is a positive-measure set $S \subseteq (0,1)^n$ such that $o(y^\star(s); z)$ is not a limit of $o(y^t(s); z)$ for all $s \in S$, there is an agent i and a positive-measure set $\hat{S} \subseteq (0,1)^n$ such that for all $s \in \hat{S}$ and all $s_i \in \hat{S}_i$, $\lim_{t \nearrow \infty} u^i(y^t(s); s_i, z) \neq u^i(y^\star(s); s_i, z)$.*

Roughly, a utility-relevant function is a mapping from actions to real numbers such that its own discontinuities imply discontinuities for some agent's utility; the formalism is necessary to deal with the question of which signal realizations are employing which actions. For example, in many auction-like models quantity allocations are utility relevant: a discontinuous change in utility in general represents a discontinuous change in utility.[35]

Lemma 12 (Continuous functions are utility-relevant)**.** *Let (W, T_W) be a topological space, and let $o : \hat{Y}^n \times \mathit{Supp}\ Z \to W$ be continuous in $y \in \hat{Y}^n$. Then o is utility-relevant.*

Theorem 4 (Equilibrium approximation)**.** *Let (W, T_W) be a topological space, and suppose that $o : \hat{Y}^n \times \mathit{Supp}\ Z \to W$ is utility-relevant. Then for almost all $s \in (0,1)^n$,*

$$\lim_{t \nearrow \infty} o\left(\alpha^t(s); z\right) = o\left(\overline{\alpha}(s); z\right).$$

I have shown that equilibrium strategies in ε^t-discretizations $\mathcal{M}^{\varepsilon^t}$ converge to an equilibrium of $\mathcal{M}$; in terms of Theorem 4 this is cast as letting the function o be the identity operator.[36]

Theorem 5 (Probabilistic approximation of observables)**.** *Let $o : \hat{Y}^n \times \mathit{Supp} Z \to \mathbb{R}^m$ be utility-relevant. Then*

$$o\left(\alpha^t(s); z\right) \xrightarrow{P} o\left(\overline{\alpha}(s); z\right).$$

Theorem 5 establishes that the base model may be empirically near its discretizations. Since models in which the underlying action space represents a continuum—or, in this case, mappings from one continuum to another—are frequently meant as approximations of discrete realities, Theorem 5 suggests that auction-like models

[35] This gets at another feature of the formalism: in action models without private information it is straightforward to construct sequences of actions at the limit of which quantity is discontinuous but utility is not (for related examples, see Reny (1999) and Jackson et al. (2002), among others). With massless private information and strictly monotone private values, these constructions go away.

[36] Reny (1999) shows that equilibria in the discretized game correspond to ε-equilibria of the base game. With type-dependent action spaces this problem is no longer well-posed, since a best response in $\mathcal{M}^{\varepsilon^t}$ may not be a feasible action in $\mathcal{M}$. If Condition 8B is satisfied, Reny's result goes through in the homogenous action space model.

with actions from a continuum may be empirically useful; in Section 5 I will use this result to show that, in most cases, equilibrium allocations and revenues also converge.[37]

5 Example: divisible-good discriminatory auctions

An auctioneer is selling $\hat{Q}$ units of a perfectly-divisible commodity to a population of $n \geq 2$ agents, $i \in \{1, \ldots, n\}$. $\hat{Q}$ is determined by a random shock $z_Q \sim \mathcal{U}(0,1)$ and market statistic p, $\hat{Q} = Q(p; z_Q)$, where Q is weakly increasing and continuous in its first argument, weakly increasing in its second argument, and $\operatorname{Supp} \hat{Q}|_p \subseteq [0, \bar{Q}]$.[38] Bidder i's private information is $s_i \sim \mathcal{U}(0,1)$; for any agent $j \neq i$, s_i and s_j are independent. Bidder i has marginal value function $v^i : [0, Q] \times (0,1) \to \mathbb{R}_+$, where $v^i(q; s_i)$ is her marginal value for the q^{th} unit of the good when her private information is s. v^i is bounded, decreasing in q, and strictly increasing and continuous in s_i.

Bidders compete for shares of the aggregate quantity $\hat{Q}$. Bidder i submits a weakly positive, weakly decreasing bid function b^i to the auctioneer, expressing the bidder's willingness to pay for the q^{th} (infinitesimal) unit. For simplicity, I consider b^i to be a function of both quantity and the agent's private information, so that $b^i(q; s)$ is agent i's submitted bid for quantity q when her signal is s. I will denote the bidder's implicit demand functions—the inverses of the bid function—by $\overline{\varphi}^i$ and $\underline{\varphi}^i$:

$$\overline{\varphi}^i(p; s) = \sup\left\{q : \ b^i(q; s) \geq p\right\}, \quad \underline{\varphi}^i(p; s) = \inf\left\{q : \ b^i(q; s) \leq p\right\}.$$

If there is no q such that $b^i(q; s) \geq p$, then $\overline{\varphi}^i(p; s) = 0$, and if there is no q such that $b^i(q; s) \leq p$, then $\underline{\varphi}^i(p; s) = \bar{Q}$; because bids are defined only on the domain of available quantities, $\overline{\varphi}^i(0; s) = \bar{Q}$.[39] Conditional on the random shock z_Q, the auctioneer compiles the submitted bid functions and computes the market-clearing price $p^\star$,

$$p^\star = \sup\left\{p : \sum_{i=1}^{n} \underline{\varphi}^i(p; s_i) \leq Q(p; z_Q) \leq \sum_{i=1}^{n} \overline{\varphi}^i(p; s_i)\right\}.$$ [40,41]

[37] Interestingly, in the discriminatory auction the same cannot be said of equilibrium prices. This is because price, unlike quantity, is an outcome statistic with no ties to incentives that are not already accounted for in revenue calculations; in particular, if bid functions are discontinuous it is possible that prices are not converging although quantities and utilities are.

[38] Note that this nests the models of constant supply, $Q(p; z_Q) = \bar{Q}$, deterministic and elastic supply, $Q(p; z_Q) = \tilde{Q}(p)$, and random inelastic supply, $Q(p; z_Q) = \tilde{Q}(z_Q)$.

[39] That $\overline{\varphi}^i(0; s) = \bar{Q}$ for all i and all s ensures that all acceptable bid functions will generate well-defined market outcomes. In particular, there is no issue with determining the proper rationed quantities if all bidders "bid c everywhere," for any constant c.

[40] Although the auctioneer does not know the agent's private information, he observes $\overline{\varphi}^i(\cdot; s_i)$ and $\underline{\varphi}^i(\cdot; s_i)$ as they are recoverable from the submitted bid function $b^i(\cdot; s_i)$.

[41] Although $p^\star$ is a function of $(b^i)_{i=1}^n$ and z_Q, for simplicity of notation I write it as its own

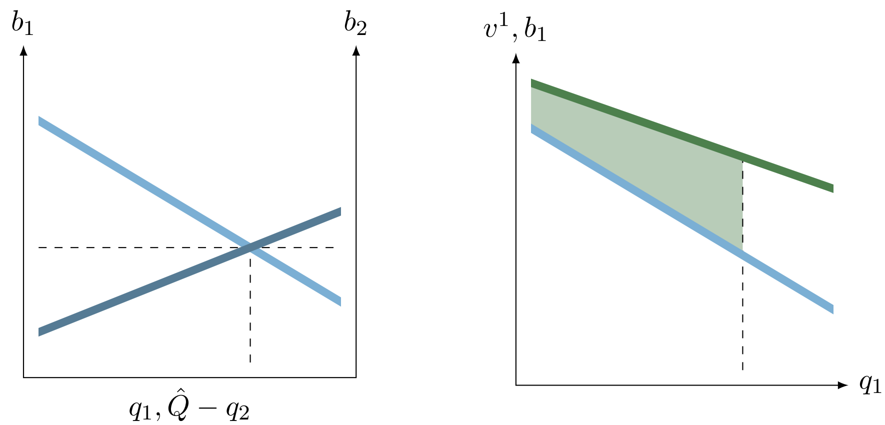

Figure 4: Market clearing and utility in the divisible-good pay-as-bid auction with two bidders and constant supply $\hat{Q} = \bar{Q}$; figure is drawn with actions and interim marginal values. The market price is determined by the inverse bids at which the sum of demand is equal to aggregate supply, and quantities are allocated accordingly. Because the seller price-discriminates, ex post utility is the area between the marginal value curve and the submitted bid curve, up to the quantity allocated.

Given this price, the auctioneer allocates to each agent her demand at this price. If $\underline{\varphi}^i(p^\star; s_i) = \overline{\varphi}^i(p^\star; s_i)$ for all i (roughly, if $b^i(\cdot; s_i)$ is strictly decreasing for each agent), then $q^i(s_1, \ldots, s_n) = \underline{\varphi}^i(p^\star; s_i)$. Otherwise, the auctioneer employs a (possibly random) tiebreaking rule,

$$q^i\left(b^i, b^{-i}; z_Q, z_q\right) \in \left[\underline{\varphi}^i\left(p^\star; s_i\right), \overline{\varphi}^i\left(p^\star; s_i\right)\right],$$
$$\sum_{i=1}^{n} q^i\left(b^i, b^{-i}; z_Q, z_q\right) = Q\left(p^\star, z_Q\right).$$

Where the tiebreaking rule is nondeterministic, randomness will be captured by z_q; henceforth let $z = (z_Q, z_q)$. Straightforward continuity arguments are sufficient to ensure that these allocation rules agree when rationing is unnecessary; as it turns out,[42] the tiebreaking rule is not essential to the existence of a pure-strategy equilibrium in the multi-unit discretization of the divisible-good model. Corollary 4 shows that this carries over to the divisible-good model itself.

I assume that the market-clearing rule implies that each agent's allocation is monotonically increasing in her own action and monotonically decreasing in those

random variable. The dependence of $p^\star$ on its inputs will be treated properly where necessary.

[42] And as also noted in, e.g., Häfner (2015).

of her opponents,

$$b'_i \geq b_i \implies q^i\left(b'_i, b_{-i}; z\right) \geq q^i\left(b'_i, b_{-i}; z\right) \ \forall z,$$
$$b'_{-i} \geq b_{-i} \implies q^i\left(b_i, b'_{-i}; z\right) \leq q^i\left(b_i, b_{-i}; z\right) \ \forall z.$$

Furthermore, where tiebreaking is necessary, I assume that the tiebreaking rule is myopic and can depend only on the discontinuity in the inverse demand function; that is, there is some $\hat{q}^i : [0, \bar{Q}]^2 \times \hat{Y}^{n-1} \times \text{Supp } Z \to [0, \bar{Q}]$ such that

$$\overline{\varphi}^i(p^\star) \neq \underline{\varphi}^i(p^\star) \implies q^i(b_i, b_{-i}; z) = \hat{q}^i\left(\overline{\varphi}^i(p^\star), \underline{\varphi}^i(p^\star), b_{-i}; z\right).$$

Conditional on the payment rule, this assumption keeps bidding incentives as local as possible; no extraneous non-local information in the bid function can be employed in tiebreaking. Most commonly-used tiebreaking rules, such as pro-rata on the margin, priority, and random allocation satisfy these assumptions. I do not pin down an explicit tiebreaking rule since it is not crucial to the existence of a pure-strategy equilibrium.

Once allocations are determined, the auctioneer employs a discriminatory payment rule,[43] and bidder i's transfer to the seller is $\int_0^{q_i} b^i(x)dx$. Her ex post utility is given by

$$u^i\left(b^i, b^{-i}; s_i, z\right) = \mathbb{E}_{z_Q}\left[\int_0^{q^i\left(b^i, b^{-i}; z\right)} v^i(x; s_i) - b^i(x)\, dx\right].$$

Let $X_D = [0, \bar{Q}]$ and $X_R = [0, \bar{b}]$, where $\bar{b} > \max_i \sup_{s_i} v^i(0; s_i)$. For an agent i with signal s_i, the feasible action space is

$$A^i(s_i) = \left\{y \in \hat{Y} : \ y \text{ decreasing, and } y \leq v^i(\cdot; s_i)\right\}.^{44}$$

That is, $A^i(s_i)$ is the set of all decreasing functions such that bids are weakly below values. Since values are weakly decreasing in quantity and $v^i(0; s_i) < \bar{b}$ for all agents i and signal realizations s_i, $A^i(s_i)$ contains all weakly positive, weakly decreasing functions that are bounded above by the agent's true marginal value.

Let $\langle \varepsilon^t \rangle_{t=1}^\infty$ be a sequence converging to $(0, 0)$, such that each $\varepsilon^t = (\varepsilon_D^t, \varepsilon_R^t)$ with $\varepsilon_R^t = [\varepsilon_D^t]^2$: for t large, the coarsening of the bid space is significantly finer than

[43]The uniform-price divisible-good auction violates Assumption 3A, since small deviations by a bidder can discretely affect payment while only marginally affecting allocation. The limiting approach of this paper can be applied to obtain equilibrium existence, but the results in this paper cannot be directly applied; see, for example, Woodward (2017).

[44]Kastl (2012) gives a model in which bidders in a uniform-price auction submit bids that are occasionally above their value functions. Similarly, Bertrand competition without private information involves submitting a constant bid "to infinity." The results here establish the existence of an equilibrium in which all bidders submit bids weakly below their value functions, but do not claim that all equilibria must exhibit this property. In related work, Pycia and Woodward (2016) show in a model without private information that all relevant bids must be weakly below values.

the coarsening of the quantity space. For each agent i, $A^{i,\varepsilon^t}(s_i)$ is the set of upper approximations of functions in $A^i(s_i)$ which are strictly decreasing,

$$A^{i,\varepsilon^t}(s_i) = \Big\{ y \in \hat{Y}^{\varepsilon^t} : y(x) > y\left(x + \varepsilon^t_D\right), \\ \text{and } \exists a_i \in A^i(s_i) \text{ s.t. } y \in \inf \Big\{ y \in \hat{Y}^{\varepsilon^t} : \; y \geq a_i \Big\} \Big\}.$$

That the bid coarsening is finer than the quantity coarsening allows strictly monotone step functions in $\mathcal{M}^{\varepsilon^t}$ to arbitrarily approximate potentially-constant bid functions in $\mathcal{M}$.

As defined, this divisible-good auction model is an auction-like model, and its ε^t-discretizations satisfy the conditions necessary to apply Theorem 1. The necessary conditions are verified in turn.

Condition 1 (Utility and actions)**.** Agent i's utility u^i is bounded, increasing, and continuous in signal since v^i is bounded, increasing, and continuous in signal. That $A^i(s_i)$ is a complete lattice is simple to verify. Let $\{b_x\}$ be a subset of $A^i(s_i)$; then for all x and q, $b_x(q) \leq v^i(q; s_i)$. Letting $\overline{b} = \sup b_x$, it follows that $\overline{b}(q) \leq v^i(q; s_i)$ for all q. Furthermore, since $b_x(q') \leq b_x(q)$ for all x whenever $q' > q$, it follows that $\overline{b}(q') \leq \overline{b}(q)$. Then $\overline{b}$ is a weakly positive monotone decreasing function that is bounded above by the agent's marginal value, and hence $\overline{b} \in A^i(s_i)$.

Condition 2 (Imitability)**.** By construction, v^i is increasing in signal; since $A^i(s_i)$ contains all weakly positive functions which are weakly below values, $s_i < s_i'$ implies $A^i(s_i) \subseteq A^i(s_i')$.

Condition 3B (Uniform upper semicontinuity (ex post))**.** By assumption of monotonicity of the allocation function, a slight increase in bid will never negatively affect quantity. Furthermore, since the difference between bids is small, any additional payment for quantities already guaranteed will also be small. If the deviation yields additional quantity, since the original bid function was below the agent's marginal value (by construction of the type-dependent action space $A^i(s_i)$), the deviation must be, at worst, only slightly above the agent's marginal value function $v^i(\cdot; s_i)$; hence any utility loss from additional quantity is small. Taken together, this implies that upward deviations cannot be discretely harmful, and similar implications hold with respect to downward deviations. This argument is formalized in Lemma 31.

Condition 4B (Local better reply availability (ex post))**.** Better reply availability is implied by Lemma 34, given in Appendix C: for any bid function b_i and $\lambda > 0$, when bidder i increases her bid from b_i to $b_i + \lambda$, bounded where appropriate by $v^i(\cdot; s_i)$, her utility in the limit is not discretely worse than her utility at the limit.

This is intuitive: as opponents' bids converge, a discrete upward shift in agent i's bid function will yield a weak increase in the quantity she is allocated. If this does not hold in the limit, the discreteness of the shift implies that her opponents' actions are not converging. Since the upward shift is near a feasible action profile,

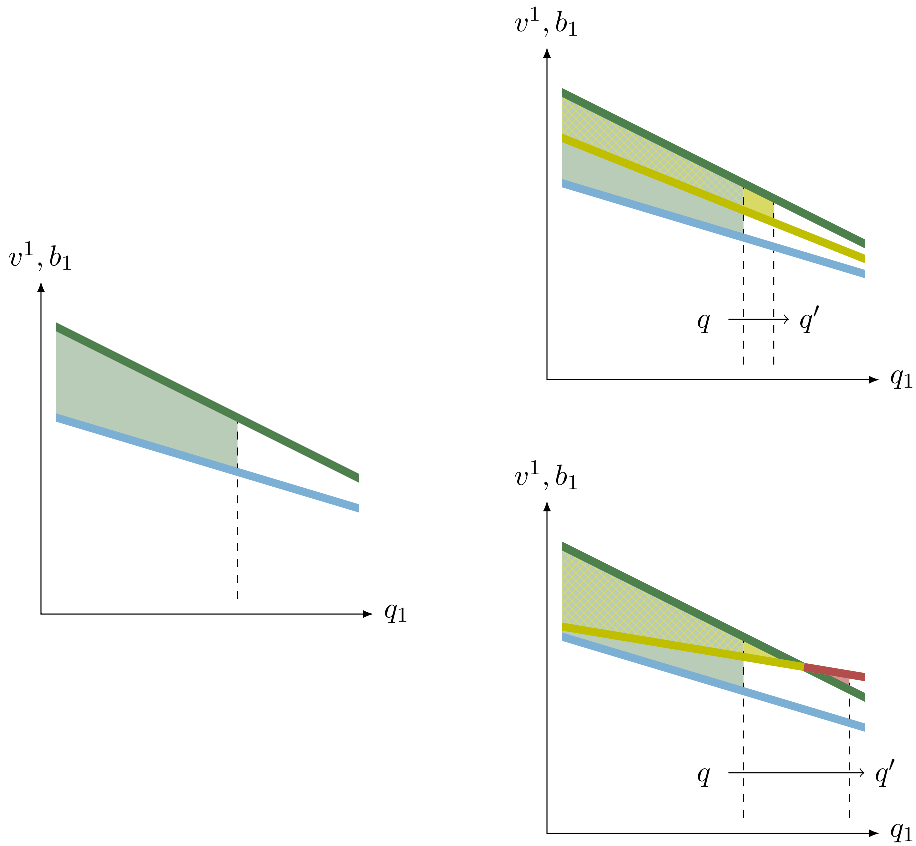

Figure 5: A small upward deviation in bid will yield weakly greater quantity. If at the new allocation the new bid has positive margins on all units, the utility impact cannot be discretely negative since the additional payment is proportional to the size of the deviation (and in general will be smaller, since typically $q_i < \bar{Q}$). If instead the new allocation yields negative margins for some units, the negative effect on utility still cannot be large, since the original bid function was weakly below values and the deviation is near the original bid function.

any losses caused by increasing her bid are commensurate to the size of the shift, by Condition 3B. This intuition is roughly illustrated in Figure 6.

Condition 5D (Surplus splitting (limiting ex post))**.** Surplus splitting is implied by market clearing. In particular, i's utility can jump down at the limit only if her allocated quantity jumps down at the limit. If this is the case for a positive-measure subset of i's types, then by market clearing there is a positive-measure set of one of her opponents who witnesses a discrete quantity increase at the limit, when facing some positive-measure set of her opponents. This is shown in Lemma 35.

Condition 6B (Single crossing and quasisupermodularity)**.** Single-crossing and quasisupermodularity are established in Lemma 32, and follow from roughly the same principles that establish these facts in the canonical single-unit first-price auction.[45]

Condition 7B (Structure of action space)**.** This follows immediately from the definition of $A^i(s_i)$. Any subset of $A^i(s_i)$ is composed of monotone functions bounded between 0 and $v^i(\cdot; s_i)$, hence the set's pointwise supremum and infimum are monotone functions bounded between 0 and $v^i(\cdot; s_i)$ and are in $A^i(s_i)$. Furthermore, if $b^\star$ is a limit of the sequence $\langle b^t \rangle_{t=1}^{\infty}$ of elements in $A^i(s_i)$, $||b^\star - v^i(\cdot; s_i) \wedge [b^\star]^+|| = 0$, and by definition $v^i(\cdot; s_i) \wedge [b^\star]^+ \in A^i(s_i)$. Then $\langle b^t \rangle_{t=1}^{\infty}$ converges to a bid in $A^i(s_i)$.

Condition 8B (Constraint super-technicality)**.** Constraint super technicality (and hence, by Lemma 2, constraint technicality) is demonstrated using the same construction as in the verification of Condition 4B: consider a bid function which is a λ-upward deviation, constrained to be no greater than value. Ex post, if this deviation yields greater quantity it incurs at most an order-λ additional cost; if it yields less quantity, monotonicity of allocation in bid implies that the bid was previously above the marginal value curve, hence reducing the allocation awarded is utility-positive, and the payment is increased by at most order-λ. In either case, nearly as much utility is obtained as from the generic bid function being analyzed, hence the agent can do nearly as well by remaining within her constraint set. This is depicted in Figure 7, and is proved in Lemma 36.

Condition 9 (Availability of discrete approximations)**.** Availability follows neatly from the construction of $X_R = [0, \bar{b}]$, where $\bar{b} > \max_i \sup_{s_i} v^i(0; s_i)$. Let $\bar{\delta} < (\bar{b} - \max_i \sup_{s_i} v^i(0; s_i))/2$, and assume that $\bar{\delta} < 1$. Then for all $\delta' \in (0, \bar{\delta})$ and all $q \in [0, \bar{Q}]$,

$$\left\lceil \frac{v^i\left(q; s_i\right)}{\left[\delta'\right]^2} \right\rceil \left[\delta'\right]^2 < \bar{b}.$$

Then for all agents i and signal realizations $s_i \in (0, 1)$, any action $y \in A^{i,(\delta', [\delta']^2)}(s_i)$ is also in $\hat{Y}$. Availability here is straightforward to satisfy, since utility is well-defined for any finite bid function, so the function space $\hat{Y}$ may be enlarged as-needed.

[45]Reny (1999) (and, by extension, McAdams (2006)) establish equilibrium existence in multi-unit pay-as-bid auctions with private information under the assumption of a continuum bid space. Since equilibrium existence in $\mathcal{M}^\varepsilon$ requires a finite bid space, the results cannot be directly applied.

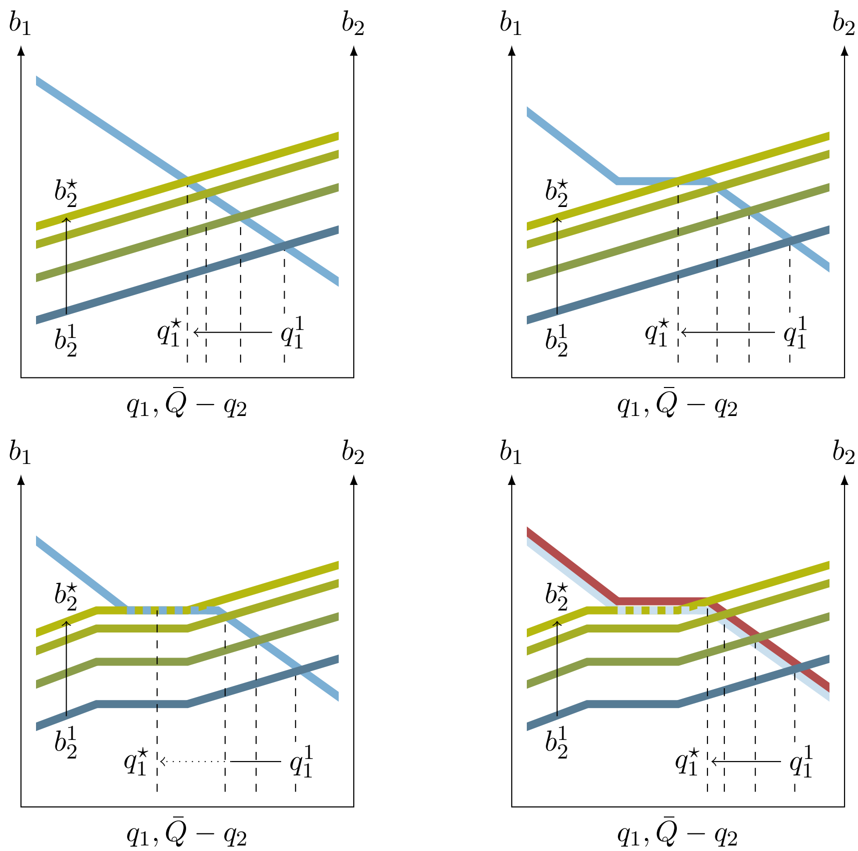

Figure 6: If agent i's utility does not converge in the limit of agent j's actions to her utility at the limit of agent j's actions, it must be that quantity is not converging; if quantity is not converging, it must be that bids are equal and along a common flat. On this interval the tiebreaking rule must be employed, hence a small upward deviation will yield a discretely greater allocation; this deviation is feasible by the assumption that utility is not converging.

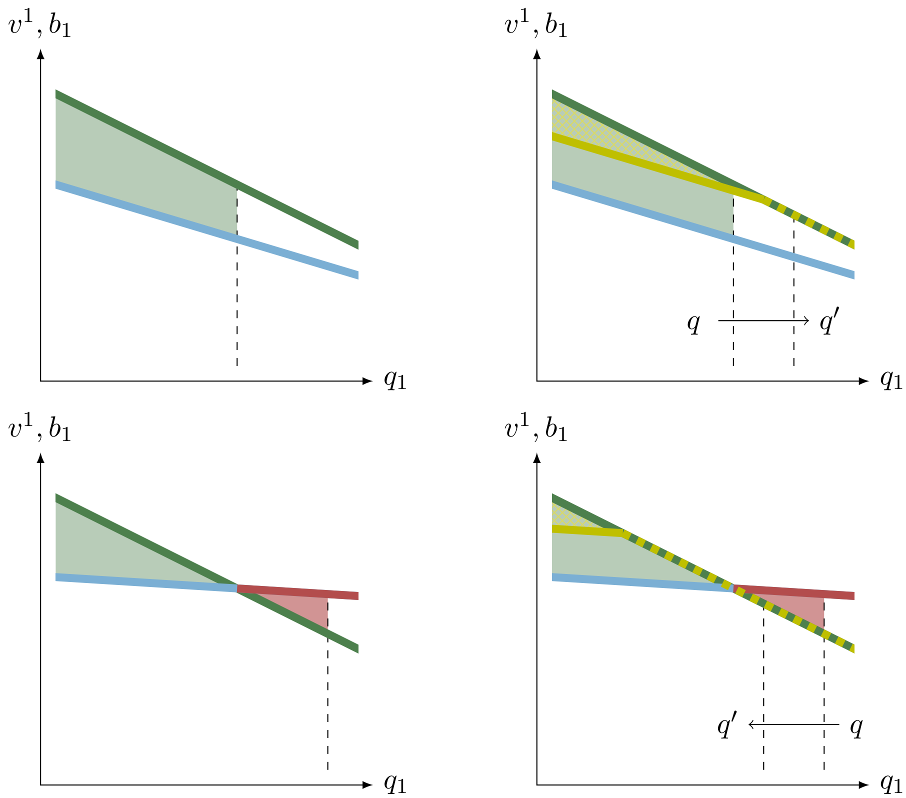

Figure 7: If $y \in \hat{Y}$ is everywhere below agent i's marginal value curve, the upward deviation $[y+\lambda] \wedge v^i(\cdot; s_i)$ yields weakly greater quantity, and since it is within $\bar{Q}\lambda$ of y incurs additional costs which are bounded above by $\bar{Q}\lambda$; then the utility of y can be approximated arbitrarily closely by this deviation. If $y \in \hat{Y}$ is somewhere above agent i's marginal value curve, the same upward deviation may yield a decrease in allocation, but only where the original bid is above marginal value, hence for quantities the agent would rather not receive; in this case, the utility of the deviation is no worse than the utility of y, minus $\bar{Q}\lambda$. The upward shift by λ is present to account for the possibility that capping bids by marginal value causes a tie to be broken against agent i's best interests.

Since the divisible-good pay-as-bid auction model with private information is an auction-like model satisfying constraint super-technicality, Corollary 4 follows directly from Theorem 2.

Corollary 4 (Equilibrium existence in pay-as-bid auctions)**.** *The divisible-good pay-as-bid auction model with private information admits a monotone pure-strategy Bayesian-Nash equilibrium in which, for each agent i, all signal realizations s_i are best-responding.*

It can be shown that allocations are utility-relevant; this immediately implies that seller revenues are also utility-relevant. Theorem 5 then implies that quantity and revenue in the ε^t-discrete auctions are approximated by quantity and revenue in the divisible-good auction.

Corollary 5 (Probabilistic convergence of observables)**.** *Let $q : \hat{Y} \times \text{Supp } Z \to \mathbb{R}^n$ and $\pi : \hat{Y} \times \text{Supp } Z \to \mathbb{R}$ represent allocations and revenue, respectively, in the divisible-good model $\mathcal{M}$. If $\langle(\beta^t)\rangle_{t=1}^{\infty}$ is a sequence of monotone pure-strategy equilibria in the ε^t-discretized models converging to the supremum-limit strategy profile $(\overline{\beta})$, then*

$$q\left(\beta^t(s);z\right) \xrightarrow{P} q\left(\overline{\beta}(s);z\right), \text{ and } \pi\left(\beta^t(s);z\right) \xrightarrow{P} \pi\left(\overline{\beta}(s);z\right).$$

Note that it is not necessarily the case that the market-clearing price is converging; this is possible because, unlike in uniform-price auctions, price is not directly strategically relevant. To see this, consider two bidders competing for deterministic and inelastic quantity $\hat{Q} = 2$, where in $\mathcal{M}^{\varepsilon^t}$ each is submitting a bid of $b(q) = 1[q \leq 1 - \varepsilon_1^t]$. Then in every discretization and in the limit each agent receives quantity $q = 1$, and in each discretization the seller's revenue is $2 - 2\varepsilon_1^t$, converging to 2. However, in every discretization the market price is 0 while at the limit it is 1.

Lastly, in the pay-as-bid auction, the equilibrium strategy profile $(\overline{\beta}^i)_{i=1}^n$ in the game in which bids must be below values is an equilibrium in the game in which bids are unconstrained.[46]

Corollary 6 (Equilibrium in unconstrained space)**.** *The supremum-limit strategy profile $(\overline{\beta}^i)_{i=1}^n$ is a pure-strategy Bayesian-Nash equilibrium in the game $\mathcal{M}'$ where $X_R = \mathbb{R}_+$, and $A^i(s_i) = \hat{Y}$ for all agents i and signal realizations s_i.*

Corollary 6 follows from Theorem 3,[47] and is fairly intuitive: if there is a value-unconstrained bid function b_i for agent i which generates strictly greater utility than

[46] This does not imply that all equilibria in the pay-as-bid auction have bids below values.

[47] There is some additional work necessary to show that there is an equilibrium with $X_R = \mathbb{R}_+$, as Theorem 3 guarantees only that equilibrium will exist with X_R compact and with a sufficiently large upper bound. In short, an extremely large bid either does not matter, in which case it is as good as bidding 0, or it does, in which case the bidder would be strictly better off bidding her value.

$\overline{\beta}^i(s_i)$, then a slight upward deviation, bounded by $v^i(\cdot; s_i)$, will "lock in" quantity obtained by rationing effects when b_i is above her value, and it will save undesirable payment when quantity would have been allocated at a bid above her value. Because the upward deviation is marginal the additional payment incurred is minor, and hence for a deviation small enough this bid will generate utility insignificantly lower than b^i. Then this slight upward deviation from b_i outperforms $\overline{\beta}^i(s_i)$ and is feasible in $A^i(s_i)$, contradicting the fact that $\overline{\beta}^i(s_i)$ is a best-response in the value-constrained auction model.

6 Conclusion

I have defined auction-like models to capture key intuitive features of auctions, and in this class of models I prove the existence of pure-strategy equilibria. This allows for the extension of known equilibrium existence results to the less-understood realm of divisible-good auction models, frequently assumed to be approximations of large multi-unit auctions. The proof approach employed demonstrates that equilibria in the divisible-good context can approximate equilibria in the discretized counterpart.

Using these tools, I prove the existence of pure-strategy equilibria in divisible-good pay-as-bid with private information, settling an outstanding question in the literature. These results permit for elastic and stochastic supply, and for risk aversion. The equilibrium constructed yields quantity allocations and seller revenues which are close, in observed probability, to those in the discretized models.

Based on the results employed and the proof approaches taken, it is likely that these results generalize to models in which types and actions are multidimensional.[48] Similarly, it is plausible that these results extend beyond the case of independent private values to models with affiliated private information. These may be productive avenues for further research.

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[48] Many auctions in which actions are finite-dimensional vectors drawn from an infinite space can be adapted so that actions meet the single-dimensionality required in this paper; for example, a bidder demanding two units can submit a monotone decreasing step function with at most two steps. It is not clear what form of auction might have actions which are two-dimensional monotone strictly-decreasing functions.

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Notational conventions for the Appendixes

Many of the results in the following Appendixes rely on arguments from convergence. Where it is not of technical importance I will assume that particular sequences converge, rather than focusing attention on convergent subsequences of arbitrary sequences.

A Proofs of alternative assumptions

Lemma 13 (Convergence of monotone functions)**.** *Let $\langle f^t \rangle_{t=1}^{\infty}$ be a sequence of functions, $f^t : (0,1) \times X_D \to X_R$, such that $f^t(w;\cdot)$ is monotone for all t and w. Suppose that $f^t(w;\cdot) \to f^\star : X_D \to X_R$ for all s; then* $\sup f^t = \inf\{f \in \hat{Y} : \ f \geq f^t\} \to f^\star$.

Proof. Suppose otherwise, and let $\bar{f}^t = \sup f^t$ and $\bar{f}^\star = \lim_{t \nearrow \infty} \bar{f}^t$. Since each f^t is monotone, $\bar{f}^t$ and hence $\bar{f}^\star$ are monotone and continuous almost everywhere. Then if $||\bar{f}^\star - f^\star|| \neq 0$, there is $\delta > 0$ and an $x \in X_D$ such that $f^\star$ is continuous at x and $\bar{f}^\star(x) > f^\star(x) + 4\delta$, and there is $\varepsilon > 0$ such that $f^\star(x') < f^\star(x) + \delta$ for all $x' \in [x, x+\varepsilon)$.

Since $\bar{f}^t \to \bar{f}^\star$, for all T there is $t > T$ such that $\bar{f}^t(x) > f^\star(x) + 3\delta$, and thus a signal $w \in (0,1)$ such that $f^t(w;x) > f^\star(x) + 2\delta$. By monotonicity, it follows that $f^t(w;x') > f^\star(x') + \delta$ for all $x' \in [x, x+\varepsilon)$, and $||f^t(s;\cdot) - f^\star|| > \varepsilon\delta$. Since ε and δ are independent of t (provided it is sufficiently large), this contradicts $f^t(w;\cdot) \to f^\star$. □

Corollary 7 (Existence of suprema)**.** *Given $y \in \hat{Y}$ and $\lambda' > 0$,* $\bar{y}^{\lambda'} = \sup D(y;\lambda')$ *is well-defined. Further, for any $\lambda > 0$ there is $\lambda' > 0$ such that $||y - \bar{y}^{\lambda'}|| < \lambda$.*

Lemma 1 (Surplus splitting implies limiting surplus splitting)**.** *Under Conditions 3A and 4A, Condition 5A implies Condition 5B.*

Proof of Lemma 1. By Lemmas 3 and 4 it is sufficient to prove this Lemma under Conditions 3B and 4B.

Let $\langle (\alpha^{k,t})_{k=1}^n \rangle_{t=1}^{\infty}$ be a sequence of strategies converging to $(\alpha^{k,\star})_{k=1}^n$, and suppose that there is an agent i and a positive-measure set $S_i \subseteq (0,1)$, and for each $s_i \in S_i$ a positive-measure set $S_{-i}(s_i)$ such that, for all $s_i \in S_i$ and $s_{-i} \in S_{-i}(s_i)$,

$$\lim_{t \nearrow \infty} U^i \left(\alpha^{i,t}(s_i), \alpha^{-i,t}(s_{-i}); s_i\right) > U^i \left(\alpha^{i,\star}(s_i), \alpha^{-i,\star}(s_{-i}); s_i\right).$$

Condition 5A implies that there is an agent j, a positive-measure set $S_j \subseteq (0,1)$, and for each $s_j \in S_j$ a positive-measure set $S_{-j}(s_j) \subseteq (0,1)^{n-1}$ such that for all $s_j \in S_j$ and all $s_{-j} \in S_{-j}(s_j)$,

$$\lim_{t \nearrow \infty} u_z^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j\right) < u_z^j \left(\alpha^{j,\star}(s_j), \alpha^{-j,\star}(s_{-j}); s_j\right).$$

By Condition 4B, for each such (s_j, s_{-j}) and any $\lambda > 0$, there is $\hat{a}_j(s_j, s_{-j}; \lambda)$ such that $||\hat{a}_j(s_j, s_{-j}; \lambda) - \alpha^{j,\star}(s_j)|| < \lambda$ and

$$\lim_{t \nearrow \infty} u_z^j \left(\hat{a}_j \left(s_j, s_{-j}; \lambda\right), \alpha^{-j,t} \left(s_{-j}\right); s_j\right) > u_z^j \left(\alpha^{j,\star} \left(s_j\right), \alpha^{-j,\star} \left(s_{-j}\right); s_j\right) - \lambda.$$

Let $\overline{a}_j(s_j; \lambda) = [\sup_{s_{-j} \in S_{-j}(s_j)} \hat{a}_j(s_j, s_{-j}; \lambda)] \vee \alpha^{j,\star}(s_j)$; since $A^j(s_j)$ is a complete lattice, $\overline{a}_j(s_j; \lambda) \in A^j(s_j)$. By Lemma 13, $\lambda \searrow 0$ implies $||\overline{a}_j(s_j; \lambda) - \alpha^{j,\star}(s_j)|| = d(s_j, \lambda) \to 0$.

By Condition 3B,

$$\begin{aligned} &u_z^j \left(\overline{a}_j \left(s_j; \lambda\right), \alpha^{-j,t} \left(s_{-j}\right); s_j\right) \\ &\quad > u_z^j \left(\alpha^{j,\star} \left(s_j\right), \alpha^{-j,\star} \left(s_{-j}\right); s_j\right) - g^j \left(\left|\left|\overline{a}_j \left(s_j; \lambda\right) - \alpha^{j,\star} \left(s_j\right)\right|\right|; s_j\right).` \end{aligned}$$

Let $\delta(s_j, s_{-j}) = u_z^j(\alpha^{j,\star}(s_j), \alpha^{-j,\star}(s_{-j}); s_j) - \lim_{t \nearrow \infty} u_z^j(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j)$. For all $s_{-j} \in S_{-j}(s_j)$, $\delta(s_j, s_{-j}) > 0$. Then

$$\begin{aligned} &\lim_{t \nearrow \infty} u_z^j \left(\overline{a}_j \left(s_j; \lambda\right), \alpha^{-j,t} \left(s_{-j}\right); s_j\right) \\ &\quad \geq u_z^j \left(\alpha^{j,\star} \left(s_j\right), \alpha^{-j,\star} \left(s_{-j}\right); s_j\right) - g^j \left(\lambda + d \left(s_j, \lambda\right); s_j\right) \\ &\quad = \lim_{t \nearrow \infty} u_z^j \left(\alpha^{j,t} \left(s_j\right), \alpha^{-j,t} \left(s_{-j}\right); s_j\right) + \delta \left(s_j, s_{-j}\right) - g^j \left(\lambda + d \left(s_j, \lambda\right); s_j\right). \end{aligned}$$

Then for λ sufficiently small,

$$\lim_{t \nearrow \infty} u_z^j \left(\overline{a}_j \left(s_j; \lambda\right), \alpha^{-j,t} \left(s_{-j}\right); s_j\right) > \lim_{t \nearrow \infty} u_z^j \left(\alpha^{j,t} \left(s_j\right), \alpha^{-j,t} \left(s_{-j}\right); s_j\right).$$

In fact, for λ sufficiently small this relation holds for a positive-measure subset $\hat{S}_j \subseteq S_j$ and $\hat{S}_{-j}(s_j) \subseteq S_{-j}(s_j)$ for all $s_j \in \hat{S}_j$. Then Condition 5B is satisfied. □

Lemma 2 (Constraint super-technicality implies constraint technicality)**.** *Condition 8B implies Condition 8A.*

Proof of Lemma 2. Let $(\alpha^j)_{j \neq i}$ be a strategy profile for agent i's opponents, and let $a_i \in A^i(s_i) \setminus \underline{A}^i(s_i)$; by definition, $a_i \in \hat{Y}$. Since u_z^i is continuous in signal, there is $s_i' < s_i$ such that

$$U^i \left(a_i, \alpha^{-i}; s_i'\right) > U^i \left(a_i, \alpha^{-i}; s_i\right) - \frac{1}{2}\lambda.$$

By Condition 8B, there is $a_i' \in A^i(s_i')$ such that

$$U^i \left(a_i', \alpha^{-i}; s_i'\right) > U^i \left(a_i, \alpha^{-i}; s_i'\right) - \frac{1}{2}\lambda.$$

Since utility is increasing in signal, it follows that

$$U^i \left(a_i', \alpha^{-i}; s_i\right) > U^i \left(a_i', \alpha^{-i}; s_i'\right) > U^i \left(a_i, \alpha^{-i}; s_i\right) - \lambda.$$

By construction, $a_i' \in \underline{A}^i(s_i)$, and Condition 8A is satisfied. □

Lemma 3 (Equivalence of uniform upper semicontinuity)**.** *Condition 3A is satisfied if and only if Condition 3B is satisfied.*

Proof of Lemma 3. Given a feasible action profile $(a_j)_{j\neq i}$ for agent i's opponents, let $(\alpha^j)_{j\neq i}$ be a strategy profile such that $\alpha^j(\cdot) = a_j$ for all $j \neq i$. Then Condition 3A implies Condition 3B.

Now, fix a strategy profile $(\alpha^j)_{j\neq i}$. Note that

$$\begin{aligned} U^i\left(\underline{a}_i, \alpha^{-i}; s_i\right) &= \mathbb{E}_{s_{-i}}\left[u_z^i\left(\underline{a}_i, \alpha^{-i}\left(s_{-i}\right); s_i\right)\right] \\ &\leq \mathbb{E}_{s_{-i}}\left[u_z^i\left(a_i, \alpha^{-i}\left(s_{-i}\right); s_i\right) + g^i\left(\left|\left|\underline{a}_i - a_i\right|\right|; s_i\right)\right] \\ &= U^i\left(a_i, \alpha^{-i}; s_i\right) + g^i\left(\left|\left|\underline{a}_i - a_i\right|\right|; s_i\right). \end{aligned}$$

The second inequality of the condition follows similarly, and Condition 3B implies Condition 3A. □

Lemma 4 (Equivalence of local better reply availability)**.** *Under Condition 3B, Condition 4B is satisfied if and only if Condition 4A is satisfied.*

Proof of Lemma 4. Given a sequence of action profiles $\langle (a^{j,t})_{j\neq i}\rangle_{t=1}^{\infty}$ converging to $(a^{j,\star})_{j\neq i}$, let $\langle (\alpha^{j,t})_{j\neq i}\rangle_{t=1}^{\infty}$ be a sequence of strategy profiles such that $\alpha^{j,t}(\cdot) = a^{j,t}$ for all (j,t), and let $(\alpha^{j,\star})_{j\neq i}$ be a strategy profile such that $\alpha^{j,\star}(\cdot) = a^{j,\star}$. Then Condition 4A implies Condition 4B.

Now, let $\lambda, \lambda' > 0$, and let $\overline{a}_i(\lambda') = \sup A^i(s_i) \cap D_{\lambda'}(a_i)$; by Condition 3B, Lemma 3, and Corollary 7, it is without loss of generality to assume that for any strategy profile $(\alpha^j)_{j\neq i}$ and any $a_i' \in A^i(s_i) \cap D_{\lambda'}(a_i)$,

$$U^i\left(\overline{a}_i\left(\lambda'\right), \alpha^{-i}; s_i\right) \geq U^i\left(a_i', \alpha^{-i}; s_i\right) - g^i\left(\lambda; s_i\right).$$

Then it follows that

$$\begin{aligned} \lim_{t\nearrow\infty} U^i\left(\overline{a}_i\left(\lambda'\right), \alpha^{-i,t}; s_i\right) &= \mathbb{E}_{s_{-i}}\left[u_z^i\left(\overline{a}_i\left(\lambda'\right), \alpha^{-i,t}\left(s_{-i}\right); s_i\right)\right] \\ &\geq \mathbb{E}_{s_{-i}}\left[u_z^i\left(a_i'\left(\alpha^{-i,t}\left(s_{-i}\right)\right), \alpha^{-i,t}\left(s_{-i}\right); s_i\right) - g^i\left(\lambda; s_i\right)\right] \\ &> U^i\left(a_i, \alpha^{-i,\star}; s_i\right) - \left[g^i\left(\lambda'; s_i\right) + g^i\left(\lambda; s_i\right)\right]. \end{aligned}$$

Since g^i goes to 0 with λ and λ', it follows that Condition 4B implies Condition 4A. □

Lemma 5 (Ex post surplus splitting implies interim surplus splitting)**.** *Condition 5C implies Condition 5A.*

Proof of Lemma 5. Suppose that $\langle (\alpha^{k,t})_{k=1}^{n}\rangle_{t=1}^{\infty}$ converges to $(\alpha^{k,\star})_{k=1}^{n}$, and that there is an agent i and a positive-measure set S_i such that for all $s_i \in S_i$,

$$\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) > U^i\left(\alpha^{i,\star}\left(s_i\right), \alpha^{-i,\star}; s_i\right).$$

Then for each s_i there is a set $S_{-i}(s_i)$ such that, for all $s_{-i} \in S_{-i}(s_i)$,

$$\lim_{t \nearrow \infty} u_z^i \left(\alpha^{i,t}(s_i), \alpha^{-i,t}(s_{-i}); s_i\right) > u_z^i \left(\alpha^{i,\star}(s_i), \alpha^{-i,\star}(s_{-i}); s_i\right).$$

By Condition 5C there is an agent j, a positive-measure set $S_j \subseteq (0,1)$, and for each $s_j \in S_j$ a positive-measure set $S_{-j}(s_j)$ such that, for all $s_j \in S_j$ and $s_{-j} \in S_{-j}(s_j)$,

$$\lim_{t \nearrow \infty} u_z^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j\right) < u_z^j \left(\alpha^{j,\star}(s_j), \alpha^{-j,\star}(s_{-j}); s_j\right).$$

Thus Condition 5A holds. □

Lemma 6 (Limiting ex post surplus splitting implies limiting interim surplus splitting)**.** *Condition 5D implies Condition 5B.*

Proof of Lemma 6. This is essentially identical to the proof of Lemma 5 above. □

Lemma 7 (Implications of action space assumptions)**.** *Let $\langle \varepsilon^t \rangle_{t=1}^{\infty}$ be a strictly decreasing sequence, $\varepsilon^t \searrow (0,0)$. Then Conditions 6B, 7B, and 9 imply Conditions 6A and 7A with respect to the ε^t-discrete models $\langle \mathcal{M}^{\varepsilon^t} \rangle_{t=1}^{\infty}$.*

Proof of Lemma 7. Let $\overline{\varepsilon}_t = \max_x \varepsilon_x^t$. First, Conditions 7B and 9 imply Condition 7A.

⊞ $A^{i,\varepsilon^t}(s_i)$ *is a lattice.* Suppose that $\boxed{a_i}, \boxed{a_i'} \in A^{i,\varepsilon^t}(s_i)$; then there are $a_i, a_i' \in A^i(s_i)$ such that

$$\boxed{a_i} = \inf\left\{a \in A^{i,\varepsilon^t}(s_i) : \ a \geq a_i\right\}, \quad \boxed{a_i'} = \inf\left\{a \in A^{i,\varepsilon^t}(s_i) : \ a \geq a_i'\right\}.$$

Then since $A^i(s_i)$ is a lattice, $\bar{a}_i = a_i \vee a_i' \in A^i(s_i)$. Suppose that there is $\boxed{y} \in A^{i,\varepsilon^t}(s_i)$ such that $\boxed{a_i} \vee \boxed{a_i'} > \boxed{y} \geq \bar{a}_i$. Then there is x such that for all $x' \in (x, x + \underline{\delta}_t)$, $\boxed{a_i}(x'), \boxed{a_i'}(x') > \boxed{y}(x') \geq a_i(x'), a_i'(x')$. Then $\boxed{a_i} \wedge \boxed{y} \geq a_i$, violating the fact that $\boxed{a_i}$ is the infimum. Condition 9 then ensures that for t sufficiently large ($\overline{\varepsilon}_t$ sufficiently small), $\boxed{a_i} \vee \boxed{a_i'} \in A^{i,\varepsilon^t}(s_i)$.

⊞ *Decreasing sequences.* Following Condition 9, for t sufficiently large and for any $a_i \in A^i(s_i)$, $\boxed{a_i}^{\varepsilon^t} \in A^{i,\varepsilon^t}(s_i)$. By construction, $||a_i - \boxed{a_i}^{\varepsilon^t}|| = O(\overline{\varepsilon}_t)$.[49] Then $\boxed{a_i}^{\varepsilon^t} \to a_i$, and $\boxed{a_i}^{\varepsilon^t} \geq a_i$ for all t.

⊞ *Convergence.* Suppose that $\langle \boxed{a_i^t} \rangle_{t=1}^{\infty}$ converges to $y_i^\star$, with $\boxed{a_i^t} \in A^{i,\varepsilon^t}(s_i)$ for all t. Then for each t, there is $a_i^t \in A^i(s_i)$ such that $||a_i^t - \boxed{a_i^t}|| = O(\overline{\varepsilon}_t)$. Then $a_i^t \to y_i^\star$, and by Condition 7B there is $a_i^\star \in A^i(s_i)$ such that $||a_i^\star - y_i^\star|| = 0$.

Next, Condition 7A and 6B imply Condition 6A. This is a direct consequence of the existence results in Reny (2011) and the finiteness of each $A^{i,\varepsilon^t}(s_i)$. □

[49]For more on this, see Lemma 17. Essentially, a_i may be approximated from above on a grid with spacing δ with error at most $(R(X_D)+R(X_R))\delta$, where $R(X) = \max X - \min X$ is independent of δ.

B Proof of Theorem 1

To begin, I define auxiliary utility functions $\hat{u}^i$ and $\hat{U}^i$,

$$\hat{u}^i\left(a_i, a_{-i}; s_i\right) = \begin{cases} u^i\left(a_i, a_{-i}; s_i\right) & \text{if } a_i \in A^i\left(s_i\right), \\ -2\left|\inf_{y \in \hat{Y}^n} u^i\left(y; s_i\right)\right| & \text{otherwise;} \end{cases}$$
$$\hat{U}^i\left(a_i, \alpha^{-i}; s_i\right) = \mathbb{E}_{s_{-i}}\left[\hat{u}^i_z\left(a_i, \alpha^{-i}\left(s_{-i}\right); s_i\right)\right].$$

These are simple methods of rendering the available action spaces uniform across types, while ensuring that (originally) infeasible actions are never selected as best responses. Similarly, in the ε-discrete model $\mathcal{M}^\varepsilon$,

$$\hat{u}^{i,\varepsilon}_z\left(a_i, a_{-i}; s_i\right) = \begin{cases} u^i_z\left(a_i, a_{-i}; s_i\right) & \text{if } a_i \in A^{i,\varepsilon}\left(s_i\right), \\ -2\left|\inf_{y \in \hat{Y}^n} u^i\left(y; s_i\right)\right| & \text{otherwise;} \end{cases}$$
$$\hat{U}^{i,\varepsilon}\left(a_i, \alpha^{-i}; s_i\right) = \mathbb{E}_{s_{-i}}\left[\hat{u}^{i,\varepsilon}_z\left(a_i, \alpha^{-i}\left(s_{-i}\right); s_i\right)\right].$$

Lemma 8 (Pure-strategy equilibrium in $\mathcal{M}^\varepsilon$)**.** *Suppose that the ε^t-discretizations of $\mathcal{M}$ satisfy Condition 2 and either Condition 6A, or Conditions 6B, 7B, and 9. Then there is T such that for all $t > T$ there is a monotone pure-strategy Bayesian-Nash equilibrium $(\alpha^{i,t})_{i=1}^n$ of the model $\mathcal{M}^{\varepsilon^t}$, in which for each agent i and almost all signals s_i,*

$$U^i\left(\alpha^i\left(s_i\right), \alpha^{-i}; s_i\right) \geq \max_{a_i \in A^{i,\varepsilon^t}\left(s_i\right)} U^i\left(a_i, \alpha^{-i}; s_i\right).$$

Proof of Lemma 8. In the case in which Condition 6A this is true by assumption; otherwise, this is a direct consequence of Lemma 7. Formally, this is existence of equilibrium with interim utility functions $\hat{U}^{i,\varepsilon^t}$, which by construction must comprise strategy profiles with actions taken from the type-dependent action spaces $A^{i,\varepsilon^t}(s_i)$. □

The proof of equilibrium existence proceeds by defining "limiting strategies," derived from equilibria of sequential refinements of the ε-discrete model $\mathcal{M}^\varepsilon$.[50] Because the proofs below frequently consider sets constrained to the rational numbers, the following shorthands will be used:

$$\mathcal{S} = (0,1) \cap \mathbb{Q}, \qquad \mathcal{S}^C = (0,1) \setminus \mathcal{S};$$
$$\mathcal{X} = X_D \cap \mathbb{Q}, \qquad \mathcal{X}^C = X_D \setminus \mathcal{Q}.$$

[50]This proof does not require that the discretized model has equal spacing, or that the limiting set of domain-range pairs is a subset of the rational plane. These are convenient assumptions for analysis and notation, and are sufficient for the present purposes. The former assumption can be replaced with an appropriate notion of uniform convergence of the approximating sets, and the latter assumption can be replaced by analyzing the Cartesian product of any countable, dense domain- and range-sets which contain the endpoints of their respective non-discretized underlying sets.

Definition 5 (Limiting strategies)**.** *Strategies* $(\alpha^{i,\square})_{i=1}^n$ *are* limiting strategies *if there exists a monotone decreasing sequence* $\langle \varepsilon^t \rangle_{t=1}^\infty$*,* $\varepsilon^t \searrow 0$*, and a sequence of equilibria of the* ε_t*-discrete model* $\mathcal{M}^{\varepsilon^t}$*,* $(\alpha^{i,\varepsilon^t})_{i=1}^n$ *such that:*

1. $\alpha^{i,\square}$ *is monotone in all arguments;*

2. *For all* $(x, s_i) \in \mathcal{X} \times \mathcal{S}$*,* $[\alpha^{i,\varepsilon^t}(s_i)](x) \to [\alpha^{i,\square}(s_i)](x)$ *pointwise.*

When coordinates are rational, limiting strategies are the natural limits of equilibria of the ε_t-discrete model. When coordinates are irrational, any values which satisfy the monotonicity constraints are permissible. Monotonicity of $\alpha^{i,\square}(s_i)$, as stated in point 1 above, is guaranteed by monotonicity of functions in $A^i(s_i)$, hence this assumption is redundant but useful for clarity, however monotonicity of $[\alpha^{i,\square}(\cdot)](x)$ is necessary: although the existence results in Reny (2011) guarantee the existence of a monotone equilibrium in $\mathcal{M}^{\varepsilon_t}$, it is possible that in some contexts a nonmonotone equilibrium will exist. The proof of existence below relies on monotonicity in both dimensions, hence point 1 is crucial.

Lemma 14 establishes that limiting strategies exist.

Lemma 14 (Existence of limiting strategies)**.** *Given any monotone decreasing sequence* $\langle \varepsilon^t \rangle_{t=1}^\infty$*, there is a subsequence* $\langle \varepsilon^{t_k} \rangle_{k=1}^\infty$*, that admits limiting strategies* $(\alpha^{i,\square})_{i=1}^n$

Proof. Lemma 8 establishes that for all t, there is a pure-strategy equilibrium $(\alpha^{i,t})$ of the ε^t-discrete model $\mathcal{M}^{\varepsilon^t}$. Selection results (Widder (1941), Theorem 16.1) imply that for any countable $\tilde{\mathcal{X}} \times \tilde{\mathcal{S}}$ there is a subsequence $\langle \varepsilon^{t_k} \rangle_{k=1}^\infty$ such that $[\alpha^{i,t_k}(s_i)](x) \to [\alpha^{i,\square}(s_i)](x)$ pointwise for all i and all $(x, s_i) \in \tilde{\mathcal{X}} \times \tilde{\mathcal{S}}$. For any such sets monotonicity of $\alpha^{i,\square}$ is guaranteed by the fact that $\alpha^{i,t}$ is monotone for all i and all ε_t. The desired result follows from letting $\tilde{\mathcal{X}} = \mathcal{X}$ and $\tilde{\mathcal{S}} = \mathcal{S}$. $\square$

Lemma 15 (L^1 convergence on $\mathcal{S}$)**.** *Let* $(\alpha^{i,\square})_{i=1}^n$ *be limiting strategies associated with some sequence* $\langle (\alpha^{i,t})_{i=1}^n \rangle_{t=1}^\infty$ *of* $\mathcal{M}^{\varepsilon^t}$*-equilibria. Then for all* i *and all* $s_i \in \mathcal{S}$*,* $||\alpha^{i,t}(s_i) - \alpha^{i,\square}(s_i)|| \to 0$.

Proof. Suppose that $\alpha^{i,\square}(s_i)$ is continuous at some $x \in \mathcal{X}$. Then for all $\lambda > 0$ there is a $\delta > 0$ such that $|[\alpha^{i,\square}(s_i)](x) - [\alpha^{i,\square}(s_i)](x + \delta')| < \lambda$ for all $\delta' \in (-\delta, \delta)$. Since $\mathcal{Q}$ is dense, there are $x_\ell, x_r \in \mathcal{X} \times (-\delta, \delta)$ such that $x_\ell < x < x_r$; by pointwise convergence of $\alpha^{i,t}$ to $\alpha^{i,\square}$ on $\mathcal{X} \times \mathcal{S}$, there is a T such that $|[\alpha^{i,t}(s_i)](x') - [\alpha^{i,\square}(s_i)](x')| < \lambda$ for $x' \in \{x_\ell, x_r\}$ and all $t > T$.

The difference between $[\alpha^{i,\square}(s_i)](x_\ell)$ and $[\alpha^{i,\square}(s_i)](x_r)$ is bounded,

$$\begin{aligned} \left|\left[\alpha^{i,\square}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x_r)\right| = &\left|\left[\alpha^{i,\square}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x)\right| \\ &+ \left|\left[\alpha^{i,\square}(s_i)\right](x) - \left[\alpha^{i,\square}(s)\right](x_r)\right| < 2\lambda. \end{aligned}$$

This implies

$$\left|\left[\alpha^{i,t}\left(s_i\right)\right]\left(x_\ell\right)-\left[\alpha^{i,t}\left(s_i\right)\right]\left(x_r\right)\right| \leq \Big[\left|\left[\alpha^{i,t}\left(s_i\right)\right]\left(x_\ell\right)-\left[\alpha^{i,\square}\left(s_i\right)\right]\left(x_\ell\right)\right|$$
$$+\left|\left[\alpha^{i,\square}\left(s_i\right)\right]\left(x_\ell\right)-\left[\alpha^{i,\square}\left(s_i\right)\right]\left(x_r\right)\right|$$
$$+\left|\left[\alpha^{i,\square}\left(s_i\right)\right]\left(x_r\right)-\left[\alpha^{i,t}\left(s_i\right)\right]\left(x_r\right)\right|\Big] < 4\lambda.$$

Since $\alpha^{i,t}(s_i)$ is monotone, this further implies that $|[\alpha^{i,t}(s_i)](x')-[\alpha^{i,t}(s_i)](x)| < 4\lambda$ for $x' \in \{x_\ell, x_r\}$. Then

$$\left|\left[\alpha^{i,t}\left(s_i\right)\right](x)-\left[\alpha^{i,\square}\left(s_i\right)\right](x)\right| \leq \Big[\left|\left[\alpha^{i,t}\left(s_i\right)\right](x)-\left[\alpha^{i,t}\left(s_i\right)\right]\left(x_\ell\right)\right|$$
$$+\left|\left[\alpha^{i,t}\left(s_i\right)\right]\left(x_\ell\right)-\left[\alpha^{i,\square}\left(s_i\right)\right]\left(x_\ell\right)\right|$$
$$+\left|\left[\alpha^{i,\square}\left(s_i\right)\right]\left(x_\ell\right)-\left[\alpha^{i,\square}\left(s_i\right)\right](x)\right|\Big] < 6\lambda.$$

Since $\lambda > 0$ may be arbitrarily small, it follows that there is T' such that $|[\alpha^{i,t}(s_i)](x)-[\alpha^{i,\square}(s_i)](x)| < \lambda$ for all $t > T'$. Then $[\alpha^{i,t}(s_i)](x) \to [\alpha^{i,\square}(s_i)](x)$ whenever $\alpha^{i,\square}(s_i)$ is continuous at x.

Since $\alpha^{i,\square}(s_i)$ is a monotone bounded function, it has at most a measure-zero set of discontinuities; hence $[\alpha^{i,t}(s_i)](x) \to [\alpha^{i,\square}(s)](x)$ for almost all x, and it is immediate that $||\alpha^{i,t}(s_i) - \alpha^{i,\square}(s_i)|| \to 0$. □

Limiting strategies are defined almost nowhere. However, since limiting strategies are monotonic in all dimensions and map into a compact space, they can be used to naturally define functions on all of $X_D \times (0,1)$.

Definition 6 (Supremum-limit strategies)**.** $(\overline{\alpha}^i)_{i=1}^n$ *is a profile of* supremum-limit *strategies if there is a profile of limiting strategies* $(\alpha^{i,\square})_{i=1}^n$ *derived from a sequence of pure-strategy equilibria* $\langle(\alpha^{i,t})_{i=1}^n\rangle_{t=1}^\infty$ *of* $\langle\mathcal{M}^{\varepsilon_t}\rangle_{t=1}^\infty$ *such that, for any* i*,*

1. *For all* $x \in \mathcal{X}$ *and* $s_i \in (0,1)$*,*

$$\left[\overline{\alpha}^i\left(s_i\right)\right](x) = \sup_{s_i' \in (0,s_i)\cap\mathcal{S}} \left[\alpha^{i,\square}\left(s_i'\right)\right](x);$$

2. *For all* $x \in \mathcal{X}^C$ *and* $s_i \in (0,1)$*,*

$$\left[\overline{\alpha}^i\left(s_i\right)\right](x) = \lim_{\delta\searrow 0} \sup_{x' \in D(x;\delta)\cap\mathcal{X}} \left[\alpha^{i,\square}\left(s_i\right)\right]\left(x'\right).$$

That is, a strategy profile $(\overline{\alpha}^i)_{i=1}^n$ is a profile of supremum-limit strategies if each strategy is left-continuous in each dimension, and it is (roughly) derived from a limiting strategy profile $(\alpha^{i,\square})_{i=1}^n$. The choice of supremum-completion in the second dimension is related to Condition 3A, and ensures that all nearby upward deviations have a positive distance from the action prescribed by the supremum-limit strategy.

In what follows, I will fix a particular convergent sequence of discretized equilibria $\langle(\alpha^{i,t})_{i=1}^n\rangle_{t=1}^\infty$, an associated limiting strategy profile $(\alpha^{i,\square})_{i=1}^n$, and an associated supremum-limit strategy profile $(\overline{\alpha}^i)_{i=1}^n$.

Lemma 16 (Almost-everywhere convergence to supremum-limit strategies)**.** *For all i and almost all s_i, $||\alpha^{i,t}(s_i) - \overline{\alpha}^i(s_i)|| \to 0$.*

Proof. Note that any limiting strategy $\alpha^{i,\square}$ has at most a measure-zero set of discontinuities (Lavrič, 1993). Let $\tilde{\alpha}^i$ be a completion of $\alpha^{i,\square}$ such that $[\tilde{\alpha}^i(s_i)](x) = [\alpha^{i,\square}(s_i)](x)$ whenever $[\alpha^{i,\square}(\cdot)](\cdot)$ is continuous at $(x; s_i)$; then $||[\alpha^{i,\square}(s_i)](x) - [\tilde{\alpha}^i(s_i)](x)|| = 0$; adapting arguments from Lemma 15 implies that $||\alpha^{i,t}(s_i) - \tilde{\alpha}^i(s_i)|| \to 0$.

Let S_δ be the set of δ-nonconverging signals s,

$$S_\delta = \left\{ s_i' : \lim_{t \nearrow \infty} \left|\left|\alpha^{i,t}\left(s_i'\right) - \tilde{\alpha}^i\left(s_i'\right)\right|\right| > \delta \right\},$$

$$\rightsquigarrow S_0 = \left\{ s_i' : \lim_{t \nearrow \infty} \left|\left|\alpha^{i,t}\left(s_i'\right) - \tilde{\alpha}^i\left(s_i'\right)\right|\right| > 0 \right\} = \bigcup_{w \in \mathbb{N}} S_{1/2^y}.$$

Letting μ^k be the Lebesgue measure on $\mathbb{R}^k$, $\mu^1(S_0) \leq \sum_{w=0}^\infty \mu^1(S_{1/2^w})$; hence if $\mu^1(S_0) > 0$ there is w such that $\mu^1(S_{1/2^w}) > 0$. Consider the measure of all points of nonconvergence,

$$\begin{aligned}
&\mu^2\left(\left\{\left(x, s_i'\right) : \left|\left[\alpha^{i,t}\left(s_i'\right)\right](x) - \left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right| \not\to 0\right\}\right) \\
&= \int_{(0,1)} \mu^1\left(\left\{x : \lim_{t \nearrow \infty} \left|\left[\alpha^{i,t}\left(s_i'\right)\right](x) - \left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right| > 0\right\}\right) d\mu^1\left(s_i'\right) \\
&\geq \int_{s_i' \in S_{1/2^w}} \mu^1\left(\left\{x : \lim_{t \nearrow \infty} \left|\left[\alpha^{i,t}\left(s_i'\right)\right](x) - \left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right| > 0\right\}\right) d\mu^1\left(s_i'\right).
\end{aligned}$$

Let $\Delta = \max X_R - \min X_R$. Note that for any $s_i' \in S_{1/2^w}$, the boundedness of A^i is sufficient to imply that[51]

$$\mu^1\left(\left\{x : \lim_{t \nearrow \infty} \left|\left[\alpha^{i,t}\left(s_i'\right)\right](x) - \left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right| > 0\right\}\right) \geq \frac{1}{2^w \Delta}.$$

Then it follows that

$$\begin{aligned}
&\mu^2\left(\left\{\left(x, s_i'\right) : \left|\left[\alpha^{i,t}\left(s_i'\right)\right](x) - \left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right| \not\to 0\right\}\right) \\
&\geq \int_{s_i' \in S_{1/2^w}} \frac{1}{2^w \max X_R} d\mu^1\left(s_i'\right) = \frac{\mu^1\left(S_{1/2^w}\right)}{2^w \Delta} > 0.
\end{aligned}$$

[51]Technically this must also include a term for the possible difference between $\alpha^{i,t}$ and the nearest element of $A^i(s_i)$. This difference is at most linear, and hence the argument does not change.

Then $\mu^2(\{(x,s_i'):|[\alpha^{i,t}(s_i')](x)-[\tilde{\alpha}^i(s_i')](x)|>0\})>0$, contradicting the fact that $||\alpha^{i,t}-\tilde{\alpha}^i||\to 0$. Since $\overline{\alpha}^i$ is a completion of $\alpha^{i,\square}$, it follows that $||\alpha^{i,t}(s_i)-\overline{\alpha}^i(s_i)||\to 0$ for almost all s_i. $\square$

Definition 7 (Upper $\mathcal{M}^\varepsilon$-approximation)**.** *An* upper $\mathcal{M}^\varepsilon$ approximation $\boxed{y}^\varepsilon$ *of an action $y\in A^i(s_i)$ is given by*

$$\boxed{y}^\varepsilon(x)\in\arg\inf_{y'\in A^{i,\varepsilon}(s_i),\ y'\geq y}\left|\left|y-y'\right|\right|.$$

Since A^{i,ε^t} is finite, an upper $\mathcal{M}^{\varepsilon^t}$ approximation of y exists (for t sufficiently large) and is in A^{i,ε^t} so long as Condition 7A is satisfied.

Henceforth, let $\bar{\varepsilon}=\max\{\varepsilon_D,\varepsilon_R\}$.

Lemma 17 (Interim utility approximation)**.** *There is a constant $C\in\mathbb{R}_+$ such that for any $y\in A^i(s_i)$,*

$$\hat{U}^{i,\varepsilon}\left(\boxed{y}^\varepsilon,\alpha^{-i};s_i\right)\geq\hat{U}^i\left(y,\alpha^{-i};s_i\right)-g^i\left(C\varepsilon;s_i\right).$$

Proof. Suppose that y is monotone increasing; by direct computation,

$$\begin{aligned}\left|\left|y-\boxed{y}^{\bar{\varepsilon}}\right|\right|&=\int_{X_D}\boxed{y}^{\bar{\varepsilon}}(x)-y(x)\,dx\\&\leq\int_{X_D}\left[y\left(\min\left\{x+\bar{\varepsilon},\sup X_D\right\}\right)+2\bar{\varepsilon}\right]-y(x)\,dx\\&=\bar{\varepsilon}y\left(\sup X_D\right)+2\left[\sup X_D-\inf X_D\right]\bar{\varepsilon}-\int_{\inf X_D}^{\inf X_D+\bar{\varepsilon}}y(x)\,dx\\&\leq\left(\left[\sup X_R-\inf X_R\right]+2\left[\sup X_D-\inf X_D\right]\right)\bar{\varepsilon}=C\bar{\varepsilon}.\end{aligned}$$

The case in which y is monotone decreasing is analogous.

By construction, $\boxed{y}^\varepsilon\in A^{i,\varepsilon}(s_i)$ and $y\in A^i(s_i)$, hence

$$\hat{U}^{i,\varepsilon}\left(\boxed{y}^\varepsilon,\alpha^{-i};s_i\right)=U^i\left(\boxed{y}^\varepsilon,\alpha^{-i};s_i\right),\text{ and }\hat{U}^{i,\varepsilon}\left(y,\alpha^{-i};s_i\right)=U^i\left(y,\alpha^{-i};s_i\right).$$

Condition 3A then implies that

$$\begin{aligned}\hat{U}^{i,\varepsilon}\left(\boxed{y}^\varepsilon,\alpha^{-i};s_i\right)&=U^i\left(\boxed{y}^\varepsilon,\alpha^{-i};s_i\right)\\&\geq U^i\left(y,\alpha^{-i};s_i\right)-g^i\left(C\bar{\varepsilon};s_i\right)\\&=\hat{U}^i\left(y,\alpha^{-i};s_i\right)-g^i\left(C\bar{\varepsilon};s_i\right).\end{aligned}$$

$\square$

Corollary 8 (Existence of utility approximation)**.** *For t sufficiently large, given any $y_i\in A^i(s_i)$, there is $\boxed{y}^{\varepsilon^t}\in A^{i,\varepsilon^t}(s_i)$ such that*

$$\hat{U}^{i,\varepsilon^t}\left(\boxed{y}^{\varepsilon^t},\alpha^{-i};s_i\right)\geq\hat{U}^i\left(y,\alpha^{-i};s_i\right)-g^i\left(C\bar{\varepsilon};s_i\right).$$

Proof. This is an immediate consequence of Lemma 17 and Condition 7A. □

Lemma 18 (Almost no upward jumps at limit)**.** *For all agents i and almost all signals $s_i \in (0,1)$,*

$$\lim_{t\nearrow\infty} \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) \geq \hat{U}^i\left(\overline{\alpha}^i(s_i), \overline{\alpha}^{-i}; s_i\right).$$

Proof. This is an application of Lemma 16, Condition 4A, and Corollary 8. Lemma 16 implies that for almost all s_i,

$$\lim_{t\nearrow 0} \left|\left|\alpha^{i,t}(s_i) - \overline{\alpha}^i(s_i)\right|\right| = 0.$$

Then it is sufficient to prove the claim of this Lemma under the assumption that agent i's action converges when her signal is s_i.

Note that in any equilibrium of $\mathcal{M}^{\varepsilon^t}$, $\hat{U}^i(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i) = U^i(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i)$. Assume then that there is $\delta > 0$ such that

$$\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}(s_i), \alpha^{-i}; s_i\right) < U^i\left(\overline{\alpha}^i(s_i), \overline{\alpha}^{-i}; s_i\right) - 3\delta.$$

By Condition 4A there is $\tilde{y}_i \in A^i(s_i)$ such that

$$\lim_{t\nearrow\infty} U^i\left(\tilde{y}_i, \alpha^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i(s_i), \overline{\alpha}^{-i}; s_i\right) - \delta > \lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) + 2\delta.$$

For any t, Lemma 17 implies that there is $\boxed{y}^{\varepsilon^t}$ such that

$$\hat{U}^{i,\varepsilon^t}\left(\boxed{y}^{\varepsilon^t}, \alpha^{-i,t}; s_i\right) \geq \hat{U}^i\left(\tilde{y}_i, \alpha^{-i,t}; s_i\right) - g^i\left(C\overline{\varepsilon}_t; s_i\right).$$

Putting these inequalities together, it follows that

$$\lim_{t\nearrow\infty} \hat{U}^{i,\varepsilon^t}\left(\boxed{y}^{\varepsilon^t}, \alpha^{-i,t}; s_i\right) + g^i\left(C\overline{\varepsilon}_t; s_i\right) > \lim_{t\nearrow\infty} \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) + 2\delta.$$

Then there is $T(s_i;\delta)$ such that for all $t > T(s_i;\delta)$,

$$\hat{U}^{i,\varepsilon^t}\left(\boxed{y}^{\varepsilon^t}, \alpha^{-i,t}; s_i\right) > \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) + \delta.$$

Since $\boxed{y}^{\varepsilon^t}$ is an action available to agent i in $\mathcal{M}^{\varepsilon^t}$, this implies that $\alpha^{i,t}(s_i)$ is not a best response.

In any $\mathcal{M}^{\varepsilon^t}$, for agent i, $\alpha^{i,t}(s_i)$ is a best response for almost all signal realizations s_i. Let $S_i(\hat{T};\delta) = \{s_i : \; T(s_i;\delta) > \hat{T}\}$; then for all $\hat{T} \in \mathbb{N}$ and $\delta > 0$, $\mu(S_i(\hat{T})) = 0$. By construction,

$$\left\{s_i : \lim_{t\nearrow\infty} \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) < \hat{U}^i\left(\overline{\alpha}^i(s_i), \overline{\alpha}^{-i}; s_i\right)\right\} = \bigcup_{\hat{T}\in\mathbb{N}} \bigcup_{w\in\mathbb{N}} S_i\left(\hat{T}; 2^{-w}\right).$$

The measure of the right-hand set is zero, and the result follows. □

Lemma 19 (Almost no downward jumps at limit)**.** *For all agents i and almost all signals $s_i \in (0,1)$,*

$$\lim_{t \nearrow \infty} \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) \leq \hat{U}^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right).$$

Proof. This follows from Lemma 18 and Conditions 3B, 4A, and 5B. Let S_i be the set of signals for which agent i's utility in the limit is strictly greater than her utility at the limit,

$$S_i = \left\{\lim_{t \nearrow \infty} \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) \leq \hat{U}^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right)\right\}.$$

Suppose that $\mu^1(S_i) > 0$, and let $S_{-i} : (0,1) \rightrightarrows (0,1)^{n-1}$ be given by

$$S_{-i}\left(s_i\right) = \left\{s_{-i} : \lim_{t \nearrow \infty} \hat{u}^{i,\varepsilon^t}\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}\left(s_{-i}\right); s_i\right) > \hat{u}^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}\left(s_{-i}\right); s_i\right)\right\}.$$

The boundedness of u^i implies that for any $s_i \in S_i$, $\mu^{n-1}(S_{-i}(s_i)) > 0$. Then by Condition 5B, there is a $\delta > 0$, an agent j, a positive-measure set $S_j \subseteq (0,1)$, for each $s_j \in S_j$ a positive-measure set $S_{-j}(s_j) \subseteq (0,1)^{n-1}$, and for any $\lambda > 0$ a sequence $\langle \hat{\alpha}^{j,t} \rangle_{t=1}^{\infty}$ with $\hat{\alpha}^{j,t}(s_j) \in A^j(s_j) \cap D_\lambda(\alpha^{j,\star}(s_j))$ such that for all $s_j \in S_j$ and $s_{-j} \in S_{-j}(s_j)$,

$$\lim_{t \nearrow \infty} u_z^j\left(\hat{\alpha}^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) \geq \lim_{t \nearrow \infty} u_z^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) + 3\delta.$$

Let $\bar{\alpha}^{j,t}(s_j) = \hat{\alpha}^{j,t}(s_j) \vee \alpha^{j,t}(s_j)$; then by Condition 3B, for all $s_{-j} \in S_{-j}(s_j)$,

$$\begin{aligned} &\lim_{t \nearrow \infty} u_z^j\left(\bar{\alpha}^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) \\ &\quad \geq \lim_{t \nearrow \infty} u_z^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) + 5\delta - g^j\left(\left|\left|\bar{\alpha}^{j,t}\left(s_j\right) - \hat{\alpha}^{j,t}\left(s_j\right)\right|\right|; s_j\right) \\ &\quad \geq \lim_{t \nearrow \infty} u_z^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) + 5\delta - g^j\left(\lambda; s_j\right). \end{aligned}$$

Then for λ sufficiently small,

$$\lim_{t \nearrow \infty} u_z^j\left(\bar{\alpha}^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) > \lim_{t \nearrow \infty} u_z^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) + 4\delta.$$

For $s_{-j} \notin S_{-j}(s_j)$,

$$\lim_{t \nearrow \infty} u_z^j\left(\bar{\alpha}^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) > \lim_{t \nearrow \infty} u_z^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) - g^j\left(\lambda; s_j\right).$$

Note that

$$\begin{aligned} U^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}; s_j\right) = {} & \Pr\left(s_{-j} \in S_{-j}\left(s_j\right)\right) \mathbb{E}_{s_{-j}}\left[u_z^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right)\right] \\ & + \Pr\left(s_{-j} \notin S_{-j}\left(s_j\right)\right) \mathbb{E}_{s_{-j}}\left[u_z^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right)\right]. \end{aligned}$$

Then

$$\begin{aligned}&\lim_{t\nearrow\infty} U^j\left(\bar{\alpha}^{j,t}(s_j), \alpha^{-j,t}; s_j\right)\\ &\quad> \lim_{t\nearrow\infty} U^j\left(\alpha^{j,t}(s_j), \alpha^{-j,t}; s_j\right)\\ &\qquad + 4\delta \Pr\left(s_{-j} \in S_{-j}(s_j)\right) - g^j(\lambda; s_j)\Pr\left(s_{-j} \notin S_{-j}(s_j)\right).\end{aligned}$$

For λ sufficiently small,

$$\lim_{t\nearrow\infty} U^j\left(\bar{\alpha}^{j,t}(s_j), \alpha^{-j,t}; s_j\right) > \lim_{t\nearrow\infty} U^j\left(\alpha^{j,t}(s_j), \alpha^{-j,t}; s_j\right) + 3\delta.$$

Appealing to Corollary 8, let $\langle \boxed{\bar{\alpha}}^{j,t}\rangle_{t=1}^{\infty}$ be a sequence of strategies with $\boxed{\bar{\alpha}}^{j,t}(s_j) \in A^{j,\varepsilon^t}(s_j)$ for all s_j such that, for t sufficiently large,

$$U^j\left(\boxed{\bar{\alpha}}^{j,t}(s_j), \alpha^{-j,t}; s_j\right) > U^j\left(\bar{\alpha}^{j,t}(s_j), \alpha^{-j,t}; s_j\right) - \delta.$$

Then

$$\lim_{t\nearrow\infty} U^j\left(\boxed{\bar{\alpha}}^{j,t}(s_j), \alpha^{-j,t}; s_j\right) > \lim_{t\nearrow\infty} U^j\left(\alpha^{j,t}(s_j), \alpha^{-j,t}; s_j\right) + 2\delta.$$

It follows that for t sufficiently large,

$$U^{j,\varepsilon^t}\left(\boxed{\bar{\alpha}}^{j,t}(s_j), \alpha^{-j,t}; s_j\right) > U^{j,\varepsilon^t}\left(\alpha^{j,t}(s_j), \alpha^{-j,t}; s_j\right) + \delta.$$

This contradicts the assumption that agent j is best responding in $\mathcal{M}^{\varepsilon^t}$ when her type is s_j; since there is a positive measure of such s_j, this contradicts the construction of equilibrium in $\mathcal{M}^{\varepsilon^t}$. □

Definition 8 (Convergent agent-types)**.** *Agent i with type $s_i \in (0,1)$ is a* convergent agent-type *if*

$$\begin{aligned}\lim_{t\nearrow\infty} \left|\left|\alpha^{i,t}(s_i) - \overline{\alpha}^i(s_i)\right|\right| = 0, \ \ \textit{and}\\ \lim_{t\nearrow\infty} \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) = \hat{U}^i\left(\overline{\alpha}^i(s_i), \overline{\alpha}^{-i}; s_i\right).\end{aligned}$$

If either of these equalities does not hold, (i, s_i) is a nonconvergent agent-type.

Lemma 20 (Utility convergence)**.** *For all agents i and almost all signals s_i, (i, s_i) is a convergent agent-type.*

Proof. For each agent i, Lemma 16 establishes that $||\alpha^{i,t}(s_i) - \overline{\alpha}^i(s_i)|| \to 0$ for almost all s_i. Lemma 18 establishes that, for almost all such agents,

$$\lim_{t\nearrow\infty} \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) \leq \hat{U}^i\left(\overline{\alpha}^i, \overline{\alpha}^{-i}; s_i\right).$$

Lemma 19 establishes that, for almost all such agents,

$$\lim_{t\nearrow\infty} \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) \geq \hat{U}^i\left(\overline{\alpha}^i, \overline{\alpha}^{-i}; s_i\right).$$

The result is then immediate. □

Lemma 21 (Best responses for convergent agent-types)**.** *For all agents i and almost all signals s_i such that (i, s_i) is a convergent agent-type, $\overline{\alpha}^i(s_i)$ is a best response to $(\overline{\alpha}^j)_{j\neq i}$.*

Proof. Suppose that agent i has a better response a_i, and that there is $\delta > 0$ with

$$\hat{U}^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > \hat{U}^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + 4\delta.$$

By Condition 4A there is $\hat{a}_i \in A^i(s_i)$ such that

$$\lim_{t\nearrow\infty} \hat{U}^i\left(\hat{a}_i, \alpha^{-i,t}; s_i\right) > \hat{U}^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) - \delta > \lim_{t\nearrow\infty} \hat{U}^{i,\varepsilon_t}\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) + 3\delta.$$

Then there is T such that for all $t > T$,

$$\hat{U}^i\left(\hat{a}_i, \alpha^{-i,t}; s_i\right) > \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) + 2\delta.$$

By Lemma 17, there is $\boxed{\hat{a}_i}^{\varepsilon^t}$ such that

$$\hat{U}^{i,\varepsilon^t}\left(\boxed{\hat{a}_i}^{\varepsilon^t}, \alpha^{-i,t}; s_i\right) > \hat{U}^{i,\varepsilon^t}\left(\hat{a}_i, \alpha^{-i,t}; s_i\right) - g^i\left(C\bar{\varepsilon}_t; \hat{a}_i, s_i\right).$$

Then there is $T' \geq T$ such that for all $t > T'$,

$$\hat{U}^{i,\varepsilon^t}\left(\boxed{\hat{a}_i}^{\varepsilon^t}, \alpha^{-i,t}; s_i\right) > \hat{U}^{i,\varepsilon^t}\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) + \delta.$$

If this holds for a positive measure of signals for agent i, this contradicts the construction of equilibrium in $\mathcal{M}^{\varepsilon^t}$. □

Lemma 22 (Best responses for nonconvergent agent-types)**.** *Under Condition 8A, $\overline{\alpha}^i(s_i)$ is a best response to $(\overline{\alpha}^j)_{j\neq i}$ for all agents i and signals $s_i \in (0,1)$.*

Proof. Suppose otherwise. Then there is $a_i \in A^i(s_i)$ and $\delta > 0$ such that

$$\hat{U}^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > \hat{U}^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + 3\delta$$

If $a_i \in \underline{A}^i(s_i)$, then there is $\gamma > 0$ such that for all $s_i' \in (s_i - \gamma, s_i)$, $a_i \in A^i(s_i')$. Since utility is increasing in signal and is upper semicontinuous in action,

$$\begin{aligned}\hat{U}^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) &> \hat{U}^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + 3\delta \\ &\geq \hat{U}^i\left(\overline{\alpha}^i\left(s_i'\right), \overline{\alpha}^{-i}; s_i\right) + 3\delta \geq \hat{U}^i\left(\overline{\alpha}^i\left(s_i'\right), \overline{\alpha}^{-i}; s_i'\right) + 3\delta.\end{aligned}$$

Since utility is continuous in signal, for γ sufficiently small it will be the case that for all $s_i' \in (s_i - \gamma, s_i)$,

$$\hat{U}^i\left(a_i, \overline{\alpha}^{-i}; s_i'\right) > \hat{U}^i\left(\overline{\alpha}^i\left(s_i'\right), \overline{\alpha}^{-i}; s_i'\right) + 2\delta.$$

This contradicts the fact that almost all such s_i' are best-responding. Then $a_i \in A^i(s_i) \setminus \underline{A}^i(s_i)$.

By Condition 8A there is $a_i' \in \underline{A}^i(s_i)$ such that

$$\hat{U}^i\left(a_i', \overline{\alpha}^{-i}; s_i\right) > \hat{U}^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) - \delta.$$

Then there is $a_i' \in \underline{A}^i(s_i)$ such that

$$\hat{U}^i\left(a_i', \overline{\alpha}^{-i}; s_i\right) > \hat{U}^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + 2\delta.$$

The rest of the proof proceeds identically to the above, implying $a_i' \in A^i(s_i) \setminus \underline{A}^i(s_i)$, a contradiction. $\square$

Theorem 3 (Unconstrained equilibrium existence)**.** *Let $(\overline{\alpha}^i)_{i=1}^n$ be a monotone pure-strategy Bayesian-Nash equilibrium of the model $\mathcal{M} = (n, u, X, A, F, Z)$ such that for all $s_i \in (0,1)$, $\alpha^i(s_i) = \sup_{s_i' < s_i} \alpha^i(s_i')$, and for almost all $s_i \in (0,1)$,*

$$U^i\left(\alpha^i\left(s_i\right), \alpha^{-i}; s_i\right) \geq \sup_{a_i \in A^i(s_i)} U^i\left(a_i, \alpha^{-i}; s_i\right).$$

Then if Condition 8B is satisfied, the strategy profile $(\alpha^i)_{i=1}^n$ constitutes a pure-strategy Bayesian-Nash equilibrium in the model $\mathcal{M}' = (n, u, X, \hat{A}, F, Z)$, where $\hat{A}^i(s_i) = \hat{Y}$, such that for each agent i and all signal realizations $s_i \in (0,1)$,

$$U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) \geq \sup_{a_i \in \hat{Y}} U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right).$$

Proof. Suppose otherwise. Then there is an agent i, a signal $s_i \in (0,1)$, an action $y \in \hat{Y}$, and a $\delta > 0$ such that

$$U^i\left(y, \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + 2\delta.$$

By Condition 8B there is $a_i \in A^i(s_i)$ such that

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > U^i\left(y, \overline{\alpha}^{-i}; s_i\right) - \delta.$$

It follows that

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + \delta.$$

This directly implies that $\overline{\alpha}^i(s_i)$ is not a best response for agent i in the constrained-action game $\mathcal{M}$ when her type is s_i. Lemma 2 demonstrates that Condition 8A is satisfied whenever Lemma 8B is satisfied, and Lemma 22 then implies that $\overline{\alpha}^i(s_i)$ is a best response in $\mathcal{M}$ for agent i when her signal is s_i, a contradiction. $\square$

C Ancillary results for assumption verification

C.1 First-price auctions (singleton domain)

Lemma 23 (Satisfaction of Condition 3B)**.** *When $A^i(s_i) \subseteq \{y : y(0) \in [0, s_i]\}$, the first-price auction model $\mathcal{M}^0$ satisfies Condition 3B.*

Proof. Checking upward deviations, for any $\lambda > 0$,

$$\begin{aligned} u_z^i\left(y, y_{-i}; s_i\right) &= \left(s_i - y\left(0\right)\right)\left(1\left[y\left(0\right) > y_{-i}\left(0\right)\right] + \frac{1}{2}1\left[y\left(0\right) = y_{-i}\left(0\right)\right]\right) \\ &\leq \left(s_i - y\left(0\right)\right)\left(1\left[y\left(0\right) + \lambda > y_{-i}\left(0\right)\right] + \frac{1}{2}1\left[y\left(0\right) + \lambda = y_{-i}\left(0\right)\right]\right) \\ &\leq u_z^i\left(y + \lambda, y_{-i}; s_i\right) + \lambda. \end{aligned}$$

Then $u_z^i(y + \lambda, y_{-i}; s_i) \geq u_z^i(y, y_{-i}; s_i) - \lambda$, establishing the upper inequality of Condition 3B.[52] The lower inequality follows similarly. □

Lemma 24 (Satisfaction of Condition 4B)**.** *When $A^i(s_i) = \{y : y(0) \in [0, s_i]\}$, the first-price auction model $\mathcal{M}^0$ satisfies Condition 4B.*

Proof. Let $\langle a_{-i,t}\rangle_{t=1}^{\infty}$ be a sequence converging to $a_{-i,\star}$, and let $a_i \in A^i(s_i)$. Consider four cases.

- There is $\delta > 0$ such that $a_i < a_{-i,\star} - 2\delta$. Then there is T such that $a_i < a_{-i,t} - \delta$ for all $t > T$, and for any $\lambda \in (0, \delta)$,

$$\lim_{t \nearrow \infty} u_z^i\left(a_i + \lambda, a_{-i,t}; s_i\right) = 0 > u_z^i\left(a_i, a_{-i,\star}; s_i\right) - \lambda.$$

- There is $\delta > 0$ such that $a_i > a_{-i,\star} + 2\delta$. Then there is T such that $a_i > a_{-i,t} + \delta$ for all $t > T$, and for any $\lambda > 0$,

$$\lim_{t \nearrow \infty} u_z^i\left(a_i, a_{-i,t}; s_i\right) = \left(s_i - a_i\left(0\right)\right) > u_z^i\left(a_i, a_{-i,\star}; s_i\right) - \lambda.$$

- $s_i > a_i = a_{-i,\star}$. Then for any $\lambda \in (0, s_i - a_i)$ there is T such that $||a_i - a_{-i,t}|| < \lambda/2$ for all $t > T$, and hence

$$\begin{aligned} \lim_{t \nearrow \infty} u_z^i\left(a_i + \lambda, a_{-i,t}; s_i\right) &= s_i - \left(a_i\left(0\right) + \lambda\right) \\ &\geq \frac{1}{2}\left(s_i - a_i\left(0\right)\right) - \lambda = u_z^i\left(a_i, a_{-i,\star}; s_i\right) - \lambda. \end{aligned}$$

- $s_i = a_i = a_{-i,\star}$. Then for any t, $u_z^i(a_i, a_{-i,t}; s_i) = 0 = u_z^i(a_i, a_{-i,\star}; s_i)$.

[52] In the main text, this is illustrated in Figure 1.

In all cases, for any $\lambda > 0$ there is an action $a_i' \in A^i(s_i)$ satisfying the necessary inequality, and Condition 4B is verified. $\square$

Lemma 25 (Satisfaction of Condition 5C)**.** *When $A^i(s_i) = \{y : y(0) \in [0, s_i]\}$, the first-price auction model $\mathcal{M}^0$ satisfies Condition 5C.*

Proof. Let $\langle(\alpha^{k,t})_{k=1}^2\rangle_{t=1}^\infty$ be a sequence of strategies converging to $(\alpha^{k,\star})_{k=1}^2$, and let S_i be the set of signal profiles such that i witnesses a discrete drop in utility at the limit,

$$S_i = \left\{(s_i, s_{-i}) : \lim_{t\nearrow\infty} u_z^i\left(\alpha^{i,t}(s_i), \alpha^{-i,t}(s_{-i}); s_i\right) > u_z^i\left(\alpha^{i,\star}(s_i), \alpha^{-i,\star}(s_{-i}); s_i\right)\right\}.$$

Since $\alpha^{i,t} \to \alpha^{i,\star}$, the only way this can occur is if, for all $(s_i, s_{-i}) \in S_i$,

$$\begin{aligned}\lim_{t\nearrow\infty} 1\left[\alpha^{i,t}(s_i) > \alpha^{-i,t}(s_{-i})\right] + \frac{1}{2}1\left[\alpha^{i,t}(s_i) = \alpha^{-i,t}(s_{-i})\right] \\ > 1\left[\alpha^{i,\star}(s_i) > \alpha^{-i,\star}(s_{-i})\right] + \frac{1}{2}1\left[\alpha^{i,\star}(v_i) = \alpha^{-i,\star}(s_{-i})\right].\end{aligned}$$

Continuity arguments imply that

$$\lim_{t\nearrow\infty} 1\left[\alpha^{i,t}(s_i) = \alpha^{-i,t}(s_{-i})\right] > 1\left[\alpha^{i,\star}(s_i) = \alpha^{-i,\star}(s_{-i})\right].$$

Market clearing implies the opposite inequalities for agent $j \neq i$. If there is a positive-measure set $\hat{S}_i \subseteq (0,1)$ such that for all $s_i \in \hat{S}_i$ there is a positive-measure $\hat{S}_{-i}(s_i) \subseteq (0,1)$ such that $\{s_i\} \times \hat{S}_{-i}(s_i) \subset S_i$, then all $s_j \in \hat{S}_{-i}(s_i)$ are such that $\alpha^{j,\star}(s_j) = \alpha^{i,\star}(s_i)$; since $\alpha^{j,\star}(s_j) \in A^j(s_j)$, almost all $s_j \in \hat{S}_{-i}(v_i)$ are such that $\alpha^{j,\star}(s_j) < s_j$.[53]

Since $\hat{S}_{-i}(s_i)$ has positive measure, there is a positive-measure S_j and a positive-measure $\hat{S}_i$ such that $\hat{S}_i \times S_j \subseteq S_i$, and $\alpha^{j,\star}(s_j) < s_j$; that is, since there is a positive measure of agent j's types submitting tied bids the set of these bids is at most countable, and since there is a positive measure of agent j's types submitting tied bids, a positive-measure set of types for bidder j is tying with positive probability against agent i. Then whenever $(s_j, s_{-j}) \in S_j \times \hat{S}_i$,

$$\begin{aligned}\lim_{t\nearrow\infty} 1\left[\alpha^{j,t}(s_j) > \alpha^{-j,t}(s_{-j})\right] + \frac{1}{2}1\left[\alpha^{j,t}(s_j) = \alpha^{-j,t}(s_{-j})\right] \\ > 1\left[\alpha^{j,\star}(s_j) > \alpha^{-j,\star}(s_{-j})\right] + \frac{1}{2}1\left[\alpha^{j,\star}(s_j) = \alpha^{-j,\star}(s_{-j})\right].\end{aligned}$$

It follows that

$$\lim_{t\nearrow\infty} u_z^j\left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j\right) < u_z^j\left(\alpha^{j,\star}(s_j), \alpha^{-j,\star}(s_{-j}); s_j\right).$$

That is, agent j benefits at the limit, and Condition 5C is verified. $\square$

[53] This is a point of some nuance in many applications. Because the value distribution is atomless, if agent i submits a bid that ties with a positive measure of agent j's bids, at most measure zero of j's types are bidding their true values.

C.2 First-price auctions (mixed strategies)

Lemma 26 (Satisfaction of Condition 3B). *The first-price auction model $\mathcal{M}^\sigma$ satisfies Condition 3B.*

Proof. As mentioned in the main text, this is equivalent to establishing uniform *lower* semicontinuity in translated actions, which are CDFs.

Let $a_i, \underline{a}_i \in \hat{Y}$ be such that $a_i \geq \underline{a}_i$ and $||a_i - \underline{a}_i|| = \lambda > 0$. Then

$$\begin{aligned}
&u_z^i\left(\underline{a}_i, a_{-i}; s_i\right) - u_z^i\left(a_i, a_{-i}; s_i\right) \\
&\quad = \int_0^2 \int_0^2 \mathbb{E}_z\left[q^i\left(b_i, b_{-i}; z\right)\right]\left(s_i - b_i\right) d\underline{a}_i\left(b_i\right) da_{-i}\left(b_{-i}\right) \\
&\qquad - \int_0^2 \int_0^2 \mathbb{E}_z\left[q^i\left(b_i, b_{-i}; z\right)\right]\left(s_i - b_i\right) da_i\left(b_i\right) da_{-i}\left(b_{-i}\right) \\
&\quad = \int_0^2 \int_0^2 \mathbb{E}_z\left[q^i\left(b_i, b_{-i}; z\right)\right]\left(s_i - b_i\right) d\left[\underline{a}_i - a_i\right]\left(b_i\right) da_{-i}\left(b_{-i}\right) \\
&\quad = \int_0^2 \int_{b_{-i}}^2 \left(s_i - b_i\right) d\left[\underline{a}_i - a_i\right]\left(b_i\right) \\
&\qquad + \frac{1}{2}\left(\Pr\left(b_i = b_{-i} \middle| \underline{a}_i\right) - \Pr\left(b_i = b_{-i} \middle| a_i\right)\right)\left(s_i - b_{-i}\right) da_{-i}\left(b_{-i}\right) \\
&\quad = \int_0^2 \int_{b_{-i}}^2 a_i\left(b_i\right) - \underline{a}_i\left(b_i\right) db_i \\
&\qquad + \left(\underline{a}_i\left(2\right) - a_i\left(2\right)\right)\left(s_i - 2\right) - \left(\underline{a}_i\left(b_{-i}\right) - a_i\left(b_{-i}\right)\right)\left(s_i - b_{-i}\right) \\
&\qquad + \frac{1}{2}\left(\Pr\left(b_i = b_{-i} \middle| \underline{a}_i\right) - \Pr\left(b_i = b_{-i} \middle| a_i\right)\right)\left(s_i - b_{-i}\right) da_{-i}\left(b_{-i}\right).
\end{aligned}$$

Since $a_i \geq \underline{a}_i$, it must be that $\int_{b_{-i}}^2 a_i(b_i) - \underline{a}_i(b_i) db_i \geq 0$; further, since $a_i(b) = 1$ for all $b \in [s_i, 2]$ and $||\underline{a}_i - a_i|| = \lambda$, it must be that $\underline{a}_i(2) - a_i(2) \geq -\lambda$. Then joining the third and fourth terms above gives

$$\begin{aligned}
&u_z^i\left(\underline{a}_i, a_{-i}; s_i\right) - u_z^i\left(a_i, a_{-i}; s_i\right) \\
&\geq \frac{1}{2}\int_0^{s_i}\left[\left(a_i\left(b_{-i}\right) - \underline{a}_i\left(b_{-i}\right)\right) + \lim_{b' \nearrow b_{-i}}\left(a_i\left(b'\right) - \underline{a}_i\left(b'\right)\right)\right]\left(s_i - b_{-i}\right) da_{-i}\left(b_{-i}\right) - 2\lambda \\
&\geq -2\lambda.
\end{aligned}$$

Then $u_z^i(\underline{a}_i, a_{-i}; s_i) \geq u_z^i(a_i, a_{-i}; s_i) - 2\lambda$, and uniform lower semicontinuity is satisfied. □

Lemma 27 (Satisfaction of Condition 4B). *The first-price auction model $\mathcal{M}^\sigma$ satisfies Condition 4B.*

Proof. Let $\langle a_{j,t}\rangle_{t=1}^{\infty}$ be a sequence of actions converging to $a_{j,\star}$, and let $a_i \in A^i(s_i)$ be a feasible action for agent i. Suppose that

$$\lim_{t\nearrow\infty} u_z^i\left(a_i, a_{j,t}; s_i\right) < u_z^i\left(a_i, a_{j,\star}; s_i\right).$$

Letting $q^i(b_i, b_{-i}; z)$ be agent i's allocation probability conditional on the realized bids (b_i, b_{-i}) and model randomness z, utility can be expressed as

$$u_z^i\left(a_i, a_{j,t}; s_i\right) = \int_0^{s_i}\int_0^{b_i} \mathbb{E}_z\left[q^i\left(b_i, b_j; z\right)\right] da_{j,t}\left(b_j\right)\left(s_i - b_i\right) da_i\left(b_i\right).$$

Because utility does not converge, there must be a positive-probability (according to distribution a_i) set of bids B_i such that for all $b_i \in B_i$,

$$\lim_{t\nearrow\infty}\left(s_i - b_i\right)\int_0^{b_i} \mathbb{E}_z\left[q^i\left(b_i, b_j; z\right)\right] da_{j,t}\left(b_j\right) < \left(s_i - b_i\right)\int_0^{b_i} \mathbb{E}_z\left[q^i\left(b_i, b_j; z\right)\right] da_{j,\star}\left(b_j\right).$$

Since this requires $s_i > b_i$, it can be restated as

$$\begin{aligned}
&\lim_{t\nearrow\infty}\left[\lim_{b\nearrow b_i} a_{j,t}\left(b\right) + \frac{1}{2}\left(a_{j,t}\left(b_i\right) - \lim_{b\nearrow b_i} a_{j,t}\left(b\right)\right)\right] \\
&= \frac{1}{2}\lim_{t\nearrow\infty}\lim_{b\nearrow b_i}\left[a_{j,t}\left(b\right) + a_{j,t}\left(b_i\right)\right] \\
&< \frac{1}{2}\lim_{b\nearrow b_i}\left[a_{j,\star}\left(b\right) + a_{j,\star}\left(b_i\right)\right] \\
&= \lim_{b\nearrow b_i} a_{j,\star}\left(b\right) + \frac{1}{2}\left(a_{j,\star}\left(b_i\right) - \lim_{b\nearrow b_i} a_{j,\star}\left(b\right)\right).
\end{aligned}$$

First, $\lim_{t\nearrow\infty}\lim_{b\nearrow b_i} a_{j,t}(b) = \lim_{b\nearrow b_i} a_{j,\star}(b)$ since $a_{j,t} \to a_{j,\star}$ and each function is monotone. Then nonconvergence of utility implies that for all $b_i \in B_i$,

$$\lim_{t\nearrow\infty} a_{j,t}\left(b_i\right) < a_{j,\star}\left(b_i\right).$$

Since $a_{j,\star}$ is monotone, there can be at most countably-many such b_i; then it is without loss of generality to assume that for all $b_i \in B_i$, $\lim_{b\nearrow b_i} a_i(b) < a_i(b_i)$: that is, each $b_i \in B_i$ is a mass point of the distribution a_i. In particular, B_i is a set of mutual mass points.

Consider an alternate action $\underline{a}_i$ defined by $\delta > 0$,

$$\underline{a}_i\left(b; \delta\right) = \begin{cases} a_i\left(0\right) & \text{if } b < \delta, \\ a_i\left(b - \delta\right) & \text{if } b \in \left[\delta, s_i\right), \\ 1 & \text{otherwise.}\end{cases}$$

Since a_i is monotone on a compact domain, for any $\lambda > 0$ there is a $\delta > 0$ such that $||\underline{a}_i(\cdot;\delta') - a_i|| < \lambda/2$ for all $\delta' < \delta$. By the arguments regarding Condition 3B,

$$u_z^i\left(\underline{a}_i\left(\cdot;\delta'\right), a_{j,t}; s_i\right) \geq u_z^i\left(a_i, a_{j,t}; s_i\right) - \lambda.$$

Since each of a_i and $a_{j,\star}$ have at most a countable number of mass points, for any $\delta > 0$ there is a $\delta' < \delta$ such that $a_i(\cdot;\delta')$ and $a_{j,\star}$ have no mass points in common (except potentially at $b_i = s_i$, which does not affect utility). By the above arguments, this implies convergence of utility and hence[54]

$$\lim_{t\nearrow\infty} u_z^i\left(\underline{a}_i\left(\cdot;\delta'\right), a_{j,t}; s_i\right) \geq u_z^i\left(\underline{a}_i\left(\cdot;\delta'\right), a_{j,\star}; s_i\right).$$

Following the arguments in Lemma 26 (Condition 3B), it must be that

$$u_z^i\left(\underline{a}_i\left(\cdot;\delta'\right), a_{j,\star}; s_i\right) > u_z^i\left(a_i, a_{j,\star}; s_i\right) - \lambda.$$

Then it follows that

$$\lim_{t\nearrow\infty} u_z^i\left(\underline{a}_i\left(\cdot;\delta'\right), a_{j,t}; s_i\right) > u_z^i\left(a_i, a_{j,\star}; s_i\right) - \lambda.$$

This verifies Condition 4B. □

Lemma 28 (Satisfaction of Condition 5C)**.** *The first-price auction model $\mathcal{M}^\sigma$ satisfies Condition 5C.*

Proof. Let $\langle(\alpha^{k,t})_{k=1}^2\rangle_{t=1}^\infty$ be a sequence of strategies converging to $(\alpha^{k,\star})_{k=1}^2$, and suppose that there is an agent i, a positive-measure set S_i, and for each $s_i \in S_i$ a positive-measure set $S_{-i}(s_i)$ such that for all $s_i \in S_i$ and $s_{-i} \in S_{-i}(s_i)$,

$$\lim_{t\nearrow\infty} u_z^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}\left(s_{-i}\right); s_i\right) > u_z^i\left(\alpha^{i,\star}\left(s_i\right), \alpha^{-i,\star}\left(s_{-i}\right); s_i\right).$$

Then there are positive-measure sets $\hat{S}_i \subseteq S_i$ and $\hat{S}_j$ such that $\hat{S}_j \subseteq S_{-i}(s_i)$ for all $s_i \in \hat{S}_i$. By the arguments employed in Lemma 27 (Condition 4B), this can occur only if, for any $s_i \in \hat{S}_i$ and $s_j \in \hat{S}_j$, $\alpha^{i,\star}(s_i)$ and $\alpha^{j,\star}(s_j)$ have a discontinuity at at least one common bid.[55] Since $\alpha^{i,\star}(s_i)$ and $\alpha^{j,\star}(s_j)$ can have at most a countable number of discontinuities, almost all $s_i \in \hat{S}_i$ and $s_j \in \hat{S}_j$ share a discontinuity at $b(s_i, s_j) < s_i, s_j$; then it is without loss of generality to assume that all s_i and s_j satisfy this property.

[54] Note that it is not necessary that the deviation eliminate common mass points: intuitively, a rightward deviation to any first-order (strictly) stochastically-dominant CDF will yield the same result. However, it is more straightforward to argue from convergence of utility, which is guaranteed by this particular deviation.

[55] This does not mean that both functions have the same set of discontinuities, only that these sets share at least one element.

Note that i's allocation probability is

$$\lim_{t\nearrow\infty}\int_0^2\int_0^2 \mathbb{E}_z\left[q^i\left(b_i,b_j;z\right)\right]d\left[\alpha^{i,t}\left(s_i\right)\right]\left(b_i\right)d\left[\alpha^{j,t}\left(s_j\right)\right]\left(b_j\right)$$
$$=\lim_{t\nearrow\infty}\int_0^2\int_0^2 \mathbb{E}_z\left[1-q^j\left(b_i,b_j;z\right)\right]d\left[\alpha^{i,t}\left(s_i\right)\right]\left(b_i\right)d\left[\alpha^{j,t}\left(s_j\right)\right]\left(b_j\right)$$
$$>\int_0^2\int_0^2 \mathbb{E}_z\left[q^i\left(b_i,b_j;z\right)\right]d\left[\alpha^{i,\star}\left(s_i\right)\right]\left(b_i\right)d\left[\alpha^{j,\star}\left(s_j\right)\right]\left(b_j\right)$$
$$=\int_0^2\int_0^2 \mathbb{E}_z\left[1-q^j\left(b_i,b_j;z\right)\right]d\left[\alpha^{i,\star}\left(s_i\right)\right]\left(b_i\right)d\left[\alpha^{j,\star}\left(s_j\right)\right]\left(b_j\right).$$

Then

$$\lim_{t\nearrow\infty}\int_0^2\int_0^2 \mathbb{E}_z\left[q^j\left(b_i,b_j;z\right)\right]d\left[\alpha^{i,t}\left(s_i\right)\right]\left(b_i\right)d\left[\alpha^{j,t}\left(s_j\right)\right]\left(b_j\right)$$
$$<\int_0^2\int_0^2 \mathbb{E}_z\left[q^j\left(b_i,b_j;z\right)\right]d\left[\alpha^{i,\star}\left(s_i\right)\right]\left(b_i\right)d\left[\alpha^{j,\star}\left(s_j\right)\right]\left(b_j\right).$$

Since this is due to a discontinuity at $b(s_i,s_j)<s_j$, it follows that for all $s_j\in\hat{S}_j$ and $s_i\in\hat{S}_i$,

$$\lim_{t\nearrow\infty}u_z^j\left(\alpha^{j,t}\left(s_j\right),\alpha^{i,t}\left(s_i\right);s_j\right)<u_z^j\left(\alpha^{j,\star}\left(s_j\right),\alpha^{i,\star}\left(s_i\right);s_j\right).$$

Then Condition 5C is verified. □

Lemma 29 (Satisfaction of Condition 6B)**.** *The first-price auction model $\mathcal{M}^\sigma$ satisfies Condition 6B.*

Proof. Suppose that $s_i<s_i'$ and $a_i>a_i'$,[56] and that

$$u_z^i\left(a_i,a_{-i};s_i\right)\leq u_z^i\left(a_i',a_{-i};s_i\right).$$

Let $q^i(b_i,b_{-i};z)$ be agent i's allocation probability. The above inequality can be rewritten as

$$\int_0^2\int_0^{b_i}\mathbb{E}_z\left[q^i\left(b_i,b_{-i};z\right)\right]\left(s_i-b_i\right)da_{-i}\left(b_{-i}\right)da_i\left(b_i\right)$$
$$\leq\int_0^2\int_0^{b_i}\mathbb{E}_z\left[q^i\left(b_i,b_{-i};z\right)\right]\left(s_i-b_i\right)da_{-i}\left(b_{-i}\right)da_i'\left(b_i\right)$$
$$\Longrightarrow s_i\int_0^2\int_0^{b_i}\mathbb{E}_z\left[q^i\left(b_i,b_{-i};z\right)\right]da_{-i}\left(b_{-i}\right)d\left[a_i\left(b_i\right)-a_i'\left(b_i\right)\right]$$
$$\leq\int_0^2\int_0^{b_i}\mathbb{E}_z\left[q^i\left(b_i,b_{-i};z\right)\right]b_i da_{-i}\left(b_{-i}\right)d\left[a_i\left(b_i\right)-a_i'\left(b_i\right)\right].$$

[56] That is, $a_i'\succeq_{\text{FOSD}} a_i$.

If the left-hand multiplicand is zero, this inequality is also satisfied for $s_i' > s_i$ and single-crossing is established; assume then that the multiplicand is nonzero. Note that

$$\int_0^2 \int_0^{b_i} \mathbb{E}_z \left[q^i \left(b_i, b_{-i}; z\right)\right] da_{-i}\left(b_{-i}\right) d\left[a_i\left(b_i\right) - a_i'\left(b_i\right)\right]$$
$$= \int_0^2 \int_{b_{-i}}^2 \mathbb{E}_z \left[q^i \left(b_i, b_{-i}; z\right)\right] d\left[a_i\left(b_i\right) - a_i'\left(b_i\right)\right] da_{-i}\left(b_{-i}\right)$$
$$\leq \frac{1}{2} \int_0^2 \left(a_i'\left(b_{-i}\right) - a_i\left(b_{-i}\right)\right) da_{-i}\left(b_{-i}\right) \leq 0.^{57}$$

Then

$$s_i' > s_i \geq \frac{\int_0^2 \int_0^{b_i} \mathbb{E}_z \left[q^i \left(b_i, b_{-i}; z\right)\right] b_i da_{-i}\left(b_{-i}\right) d\left[a_i\left(b_i\right) - a_i'\left(b_i\right)\right]}{\int_0^2 \int_0^{b_i} \mathbb{E}_z \left[q^i \left(b_i, b_{-i}; z\right)\right] da_{-i}\left(b_{-i}\right) d\left[a_i\left(b_i\right) - a_i'\left(b_i\right)\right]}.$$

Then single-crossing is satisfied; strict single-crossing follows similarly.

Now suppose that a_i and a_i' are such that

$$u_z^i\left(a_i \wedge a_i', a_{-i}; s_i\right) \geq u_z^i\left(a_i, a_{-i}; s_i\right).$$

Since for any b, $a_i(b) + a_i'(b) = [a_i \wedge a_i'](b) + [a_i \vee a_i'](b)$, appealing to the expected utility expressions above it follows that

$$u_z^i\left(a_i \vee a_i', a_{-i}; s_i\right) = u_z^i\left(a_i, a_{-i}; s_i\right) + u_z^i\left(a_i', a_{-i}; s_i\right) - u_z^i\left(a_i \wedge a_i', a_{-i}; s_i\right).$$

It is immediate that

$$u_z^i\left(a_i \vee a_i', a_{-i}; s_i\right) \leq u_z^i\left(a_i', a_{-i}; s_i\right).$$

Then quasisubmodularity is satisfied;[58] strict quasisubmodularity follows similarly. □

C.3 Divisible-good auctions

Lemma 30 (Market outcome monotonicity)**.** *Let b_i', b_i be bids for agent i, $b_i' \geq b_i$, and let $(b_j)_{j \neq i}$ be bids for agent i's opponents. For any $z \in Supp\ Z$,*

$$p^\star\left(b_i', b_{-i}; z\right) \geq p^\star\left(b_i, b_{-i}; z\right), \text{ and } Q\left(p^\star\left(b_i', b_{-i}; z\right); z\right) \geq Q\left(p^\star\left(b_i, b_{-i}; z\right); z\right).$$

Proof. The latter claim follows immediately from the former, since Q is monotonically increasing in p, so it suffices to establish the former inequality. Suppose that $p' = p^\star(b_i', b_{-i}; z) < p^\star(b_i, b_{-i}; z) = p$. Then $Q(p'; z) \leq Q(p; z)$. Since allocations are

[58] Since analysis here is in strategies which have been passed through a monotone decreasing "translation function," Condition 6B in this context requires quasi*sub*modularity rather than quasi*super*modularity.

weakly increasing in own-action, $q^i(b_i', b_{-i}; z) \geq q^i(b_i, b_{-i}; z)$; since allocations are weakly decreasing in price, $q^j(b_j, b_i', b_{-ij}; z) \geq q^j(b_j, b_i, b_{-ij}; z)$ for all $j \neq i$. Then by market clearing, it follows that

$$q^i\left(b_i', b_{-i}; z\right) = q^i\left(b_i, b_{-i}; z\right), \text{ and } q^j\left(b_j, b_i', b_{-ij}; z\right) = q^j\left(b_j, b_i, b_{-ij}; z\right) \; \forall j \neq i.$$

Since $b_i' \geq b_i$, it follows that

$$\begin{aligned} \underline{\varphi}_i(p) + \sum_{j \neq i} \underline{\varphi}_j(p) &\leq \underline{\varphi}_i'(p) + \sum_{j \neq i} \underline{\varphi}_j(p), \text{ and} \\ \overline{\varphi}_i(p) + \sum_{j \neq i} \overline{\varphi}_j(p) &\leq \overline{\varphi}_i'(p) + \sum_{j \neq i} \overline{\varphi}_j(p). \end{aligned}$$

Since $Q(p; z) = Q(p'; z)$, it follows that $p' \geq p$, a contradiction. Then it must be that $p^\star(b_i', b_{-i}; z) \geq p^\star(b_i, b_{-i}; z)$. □

Lemma 31 (Satisfaction of Condition 3B.)**.** *The divisible-good pay-as-bid auction model $\mathcal{M}$ satisfies Condition 3B.*

Proof. Let $b_i \in A^i(s_i)$, $(b_j)_{j \neq i}$ be strategies for agent i's opponents, and $\overline{y}_i \in \hat{Y}$, $\overline{y}_i \geq b_i$; define $\lambda = ||\overline{y}_i - b_i||$. Then conditional on model uncertainty $z = (z_Q, z_q)$, the difference in ex post utility is given by

$$\begin{aligned} &u^i\left(b_i, b_{-i}; s_i, z\right) - u^i\left(\overline{y}_i, b_{-i}; s_i, z\right) \\ &= \int_0^{q^i(b_i, b_{-i}; z)} v^i\left(x; s_i\right) - b_i\left(x\right) dx - \int_0^{q^i(\overline{y}_i, b_{-i}; z)} v^i\left(x; s_i\right) - \overline{y}_i\left(x\right) dx \\ &= -\left(\int_0^{q^i(b_i, b_{-i}; z)} b_i\left(x\right) dx - \int_0^{q^i(\overline{y}_i, b_{-i}; z)} \overline{y}_i\left(x\right) dx\right) + \int_{q^i(\overline{y}_i, b_{-i}; z)}^{q^i(b_i, b_{-i}; z)} v^i\left(x; s_i\right) dx \\ &= -\int_0^{q^i(\overline{y}_i, b_{-i}; z)} b_i\left(x\right) - \overline{y}_i\left(x\right) dx - \int_{q^i(b_i, b_{-i}; z)}^{q^i(\overline{y}_i, b_{-i}; z)} v^i\left(x; s_i\right) - b_i\left(x\right) dx \\ &\leq \int_0^{q^i(\overline{y}_i, b_{-i}; z)} \overline{y}_i\left(x\right) - b_i\left(x\right) dx \leq \lambda. \end{aligned}$$

These inequalities follow from the assumption that the allocation rule is monotone in the bid submitted, that $\overline{y}_i \geq b_i$, and that $b_i \leq v^i(\cdot; s_i)$. Then in expectation over model randomness,

$$\begin{aligned} u_z^i\left(\overline{y}_i, b_{-i}; s_i\right) &= \mathbb{E}_z\left[\int_0^{q^i(\overline{y}_i, b_{-i}; z)} v^i\left(x; s_i\right) - \overline{y}_i\left(x\right) dx\right] \\ &\geq \mathbb{E}_z\left[\int_0^{q^i(b_i, b_{-i}; z)} v^i\left(x; s_i\right) - b_i\left(x\right) dx - \lambda\right] \\ &= u_z^i\left(b_i, b_{-i}; s_i\right) - \lambda. \end{aligned}$$

Then Condition 3B is satisfied. □

Lemma 32 (Quasisupermodularity and single crossing in $\mathcal{M}^{\varepsilon}$)**.** *If $b_i, b_i' \in \hat{Y}^{\varepsilon}$, then*

$$U^i\left(b_i, \beta^{-i}; s_i\right) \geq (>)\, U^i\left(b_i \wedge b_i', \beta^{-i}; s_i\right) \implies U^i\left(b_i \vee b_i', \beta^{-i}; s_i\right) \geq (>)\, U^i\left(b_i', \beta^{-i}; s_i\right).$$

Furthermore, if $b_i \leq b_i'$ and $s_i \leq s_i'$,

$$U^i\left(a_i', \beta^{-i}; s_i\right) \geq (>)\, U^i\left(b_i, \beta^{-i}; s_i\right) \implies U^i\left(b_i', \beta^{-i}; s_i'\right) \geq (>)\, U^i\left(b_i, \beta^{-i}; s_i'\right).$$

Proof. To establish single crossing, let $b_i \leq b_i'$ and $s_i \leq s_i'$. Assume that

$$\begin{aligned} U^i\left(b_i', \beta^{-i}; s_i\right) &= \mathbb{E}_{s_{-i},z}\left[\int_0^{q^i\left(b_i', \beta^{-i}(s_{-i}); z\right)} v^i\left(x; s_i\right) - b_i'\left(x\right) dx\right] \\ &\geq \mathbb{E}_{s_{-i},z}\left[\int_0^{q^i\left(b_i, \beta^{-i}(s_{-i}); z\right)} v^i\left(x; s_i\right) - b_i\left(x\right) dx\right] = U^i\left(b_i, \beta^{-i}; s_i\right). \end{aligned}$$

The inequality can be rearranged to see

$$\begin{aligned} &\mathbb{E}_{s_{-i},z}\left[\int_{q^i\left(b_i, \beta^{-i}(s_{-i}); z\right)}^{q^i\left(b_i', \beta^{-i}(s_{-i}); z\right)} v^i\left(x; s_i\right) dx\right] \\ &\quad \geq \mathbb{E}_{s_{-i},z}\left[\int_0^{q^i\left(b_i', \beta^{-i}(s_{-i}); z\right)} b_i'\left(x\right) dx - \int_0^{q^i\left(b_i, \beta^{-i}(s_{-i}); z\right)} b_i\left(x\right) dx\right]. \end{aligned}$$

Since allocations are increasing in bid and $b_i' \geq b_i$, the fact that $v^i(x; \cdot)$ is monotone increasing implies that

$$\begin{aligned} &\mathbb{E}_{s_{-i},z}\left[\int_{q^i\left(b_i, \beta^{-i}(s_{-i}); z\right)}^{q^i\left(b_i', \beta^{-i}(s_{-i}); z\right)} v^i\left(x; s_i'\right) dx\right] \\ &\quad \geq \mathbb{E}_{s_{-i},z}\left[\int_{q^i\left(b_i, \beta^{-i}(s_{-i}); z\right)}^{q^i\left(b_i', \beta^{-i}(s_{-i}); z\right)} v^i\left(x; s_i\right) dx\right] \\ &\quad \geq \mathbb{E}_{s_{-i},z}\left[\int_0^{q^i\left(b_i', \beta^{-i}(s_{-i}); z\right)} b_i'\left(x\right) dx - \int_0^{q^i\left(b_i, \beta^{-i}(s_{-i}); z\right)} b_i\left(x\right) dx\right]. \end{aligned}$$

Then single crossing is satisfied; strict single crossing follows similarly.

To establish quasisupermodularity, let $\overline{b}_i = b_i \vee b_i'$ and $\underline{b}_i = b_i \wedge b_i'$, and let $G, G', \overline{G}, \underline{G}$ be the distributions of agent i's allocation under actions $b_i, b_i', \overline{b}_i, \underline{b}_i$, respectively. Compare

$$\begin{aligned}
&U^i\left(b_i,\beta^{-i};s_i\right)+U^i\left(b_i',\beta^{-i};s_i\right)\\
&=\int_0^{\bar{Q}}\int_0^q v^i\left(x;s_i\right)-b_i\left(x\right)dxdG\left(q\right)+\int_0^{\bar{Q}}\int_0^q v^i\left(x;s_i\right)-b_i'\left(x\right)dxdG'\left(q\right)\\
&=\int_0^{\bar{Q}}\int_0^q v^i\left(x;s_i\right)-b_i\left(x\right)dxd\left[G\left(q\right)+G'\left(q\right)\right]+\int_0^{\bar{Q}}\int_0^q \left(b_i\left(x\right)-b_i'\left(x\right)\right)dxdG'\left(q\right)\\
&=\int_0^{\bar{Q}}\int_0^q v^i\left(x;s_i\right)-b_i\left(x\right)dxd\left[\overline{G}\left(q\right)+\underline{G}\left(q\right)\right]+\int_0^{\bar{Q}}\int_0^q \left(b_i\left(x\right)-b_i'\left(x\right)\right)dxdG'\left(q\right)\\
&=\int_0^{\bar{Q}}\int_0^q v^i\left(x;s_i\right)-\overline{b}_i\left(x\right)dxd\overline{G}\left(q\right)+\int_0^{\bar{Q}}\int_0^q v^i\left(x;s_i\right)-\underline{b}_i\left(x\right)dxd\underline{G}\left(q\right)\\
&\qquad+\int_0^{\bar{Q}}\int_0^q \overline{b}_i\left(x\right)dxd\overline{G}\left(q\right)+\int_0^{\bar{Q}}\int_0^q \underline{b}_i\left(x\right)dxd\underline{G}\left(q\right)\\
&\qquad-\int_0^{\bar{Q}}\int_0^q b_i\left(x\right)dxdG\left(q\right)-\int_0^{\bar{Q}}\int_0^q b_i'\left(x\right)dxdG'\left(q\right).
\end{aligned}$$

That $\overline{G}+\underline{G}=G'+G$ follows from the constraint that actions in $A^{i,\varepsilon}(s_i)$ are strictly decreasing between steps, hence when tiebreaking is necessary the endpoints of the flat interval of the bid function are independent of the bid function itself; assumed myopia of the tiebreaking rule then implies this equality.

In the above, the first two terms together are $U^i(\overline{b}_i,\beta^{-i};s_i)+U^i(\underline{b}_i,\beta^{-i};s_i)$; then if the remaining four terms are weakly negative, quasisupermodularity will hold. Let $\ell(q)=\{\tilde{q}\leq q:\ a_i(\tilde{q})\leq a_i'(\tilde{q})\}$, and let $\ell'(q)=[0,q]\setminus\ell(q)$. Note that

$$\int_0^{\bar{Q}}\int_0^q \overline{b}_i\left(x\right)dxd\overline{G}\left(q\right)=\int_0^{\bar{Q}}\int_{\ell(q)} b_i'\left(x\right)dx+\int_{\ell'(q)} b_i\left(x\right)dxd\overline{G}\left(q\right).$$

Similar equations hold for the other bid functions and their associated distributions. It follows that

$$\begin{aligned}
&\int_0^{\bar{Q}}\int_0^q \overline{b}_i\left(x\right)dxd\overline{G}\left(q\right)-\int_0^{\bar{Q}}\int_0^q b_i\left(x\right)dxdG\left(q\right)\\
&\quad=\int_0^{\bar{Q}}\int_{\ell'(q)} b_i\left(x\right)dxd\left[\overline{G}\left(q\right)-G\left(q\right)\right]+\int_0^{\bar{Q}}\int_{\ell(q)} b_i'\left(x\right)dxd\overline{G}\left(q\right)\\
&\qquad-\int_0^{\bar{Q}}\int_{\ell(q)} b_i\left(x\right)dxdG\left(q\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^{\bar{Q}}\int_0^q \underline{b}_i(x)\,dxd\underline{G}(q) - \int_0^{\bar{Q}}\int_0^q b_i'(x)\,dxdG'(q) \\
&\quad = \int_0^{\bar{Q}}\int_{\ell'(q)} b_i'(x)\,dxd\left[\underline{G}(q) - G'(q)\right] + \int_0^{\bar{Q}}\int_{\ell(q)} b_i(x)\,dxd\underline{G}(q) \\
&\qquad - \int_0^{\bar{Q}}\int_{\ell(q)} b_i'(x)\,dxdG'(q).
\end{aligned}$$

Note that $G+G' = \overline{G}+\underline{G}$ implies $\overline{G}-G = G'-\underline{G} \equiv \Delta G$; then the final four terms sum to

$$\begin{aligned}
&\int_0^{\bar{Q}}\int_{\ell'(q)} b_i(x) - b_i'(x)\,dxd\Delta G(q) \\
&\quad + \int_0^{\bar{Q}}\int_{\ell(q)} b_i'(x)\,dxd\overline{G}(q) - \int_0^{\bar{Q}}\int_{\ell(q)} b_i(x)\,dxdG(q) \\
&\qquad + \int_0^{\bar{Q}}\int_{\ell(q)} b_i(x)\,dxd\underline{G}(q) - \int_0^{\bar{Q}}\int_{\ell(q)} b_i'(x)\,dxdG'(q) \\
&= \int_0^{\bar{Q}}\int_{\ell'(q)} b_i(x) - b_i'(x)\,dxd\Delta G(q) \\
&\qquad + \int_0^{\bar{Q}}\int_{\ell(q)} b_i'(x)\,dxd\hat{\Delta}G(q) - \int_0^{\bar{Q}}\int_{\ell(q)} b_i(x)\,dxd\hat{\Delta}G(q) \\
&= \int_0^{\bar{q}}\int_{\ell'(q)} b_i(x) - b_i'(x)\,dxd\Delta G(q) \\
&\qquad + \int_0^{\bar{Q}}\int_{\ell(q)} b_i'(x) - b_i(x)\,dxd\hat{\Delta}G(q).
\end{aligned}$$

Note that when $d\Delta G$ is nonzero, the inner integrand is locally constant: it must be that at almost all such q, $\overline{G}(q) \neq G(q)$, and hence $\bar{b}_i(q) = b_i'(q) \geq b_i(q)$, so $q \notin \ell'(\bar{Q})$. Since across any such interval $\int d\Delta G(q) = 0$, it follows that the first term in the above is zero; similar logic implies the same for the second term.

Then the final four terms in the summation cancel, leaving

$$U^i\left(b_i, \beta^{-i}; s_i\right) + U^i\left(b_i', \beta^{-i}; s_i\right) = U^i\left(\bar{b}_i, \beta^{-i}; s_i\right) + U^i\left(\underline{b}_i, \alpha^{-i}; s_i\right).$$

Then $U^i(b_i, \beta^{-i}; s_i) \geq U^i(\underline{b}_i, \beta^{-i}; s_i)$ implies $U^i(\bar{b}_i, \beta^{-i}; s_i) \geq U^i(b_i', \beta^{-i}; s_i)$, and quasisupermodularity is satisfied; strict quasisupermodularity follows similarly.

□

To simplify notation in the following, let $I : \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ be given by $I(a,b) = (\min\{a,b\}, \max\{a,b\})$.

Lemma 33 (Discontinuous allocations)**.** *Let* $\langle (b_{i,t})_{i=1}^{n} \rangle_{t=1}^{\infty}$ *be a sequence of bid functions converging to* $(b_{i,\star})_{i=1}^{n}$*. Suppose that* $\lim_{t \nearrow \infty} q^i(b_{i,t}, b_{-i,t}; z) \neq q^i(b_{i,\star}, b_{-i,\star}; z)$*, and that the limit exists. Then*

$$b_{i,\star}\left(q'\right) = b_{i,\star}\left(q''\right) = p^{\star,\star},\ \forall q', q'' \in I\left(\lim_{t \nearrow \infty} q^i\left(b_{i,t}, b_{-i,t}; z\right), q^i\left(b_{i,\star}, b_{-i,\star}; z\right)\right).$$

Proof. For any agent j, define quantities

$$q_{j,t} = q^j\left(b_j, b_{-j,t}; z\right),\ q_{j,\star} = q^j\left(b_j, b_{-j,\star}; z\right),\ \overline{q}_j = \lim_{t \nearrow \infty} q_{j,t}.$$

Assume without loss of generality that $\overline{q}_i < q_{i,\star}$; by market clearing there is $J \neq \emptyset$ such that for all $j \in J$, $\overline{q}_j > q_{j,\star}$. Let $\delta > 0$ be such that $\lim_{t \nearrow \infty} |q_{k,t} - q_{k,\star}| > 2\delta$ for all $k \in J \cup \{i\}$.

By market clearing, it must be that for all t sufficiently large and all $j \in J$,

$$b_{j,t}\left(q_{j,\star} + \delta\right) \geq b_{i,t}\left(q_{i,\star} - \delta\right),\ \text{and } b_{j,t}\left(\overline{q}_j - \delta\right) \geq b_{i,t}\left(\overline{q}_i + \delta\right).$$

In the limit, it must be that

$$\begin{aligned} \lim_{t \nearrow \infty} b_{j,t}\left(q_{j,\star} + \delta\right) &\geq b_{j,\star}\left(\overline{q}_j - \delta\right), \\ \text{and } \lim_{t \nearrow \infty} b_{i,t}\left(q_{i,\star} - \delta\right) &\leq b_{i,\star}\left(\overline{q}_i + \delta\right). \end{aligned}$$

It follows that

$$\begin{aligned} \lim_{t \nearrow \infty} b_{j,t}\left(q_{j,\star} + \delta\right) &\geq \lim_{t \nearrow \infty} b_{j,t}\left(\overline{q}_j - \delta\right) \\ &\geq \lim_{t \nearrow \infty} b_{i,t}\left(\overline{q}_i + \delta\right) \geq \lim_{t \nearrow \infty} b_{i,t}\left(q_{i,\star} - \delta\right). \end{aligned} \tag{1}$$

Further, it must be the case that $\lim_{t \nearrow \infty} b_{j,t}(q_{j,\star} + \delta) \leq \lim_{t \nearrow \infty} b_{i,t}(q_{i,\star} - \delta)$. Otherwise, monotonicity and convergence together imply that

$$b_{j,\star}\left(q_{j,\star} + \delta'\right) > b_{i,\star}\left(q_{i,\star} - \delta'\right)\ \forall \delta' \in (0, \delta).$$

This contradicts the definition of $q_{j,\star}$ and $q_{i,\star}$. Then the inequalities in (1) hold with equality, and

$$\lim_{t \nearrow \infty} b_{i,t}\left(q_{i,\star} - \delta\right) = b_{i,\star}\left(\overline{q}_i + \delta\right).$$

Since this is true for all $\delta' \in (0, \delta)$, it follows that

$$\lim_{t \nearrow \infty} b_{i,t}\left(\lim_{q' \nearrow q_{i,\star}} q'\right) = b_{i,\star}\left(\lim_{q' \searrow \overline{q}_i} q'\right).$$

Then monotonicity and convergence imply that

$$b_{i,\star}\left(q'\right) = b_{i,\star}\left(q''\right),\ \forall q', q'' \in I\left(\lim_{t \nearrow \infty} q^i\left(b_{i,t}, b_{-i,t}; z\right), q^i\left(b_{i,\star}, b_{-i,\star}; z\right)\right).$$

Regarding the second equality, note that since agent i's bid is locally flat,

$$p^{\star,\star} > \lim_{q' \searrow \bar{q}_i} b_{i,\star}\left(q'\right) \implies q_{i,\star} \leq \bar{q}_i.$$

Further, the above proof implies that

$$\lim_{t \nearrow \infty} b_{j,t}\left(q_{j,\star} + \delta\right) = b_{j,\star}\left(\bar{q}_j - \delta\right).$$

Working through a similar set of arguments, it follows that

$$p^{\star,\star} < \lim_{q' \searrow \bar{q}_i} b_{i,\star}\left(q'\right) \implies q_{j,\star} \geq \bar{q}_j.$$

These are both contradictions, hence

$$p^{\star,\star} = \lim_{q' \searrow \bar{q}_i} b_{i,\star}\left(q'\right).$$

□

Lemma 34 (Utility dominance in limit)**.** *Suppose that $b_i \in A^i(s_i)$ is a feasible bid function for agent i when her signal is s_i, and that $\langle (b_{j,t})_{j \neq i} \rangle_{t=1}^{\infty}$ are bid functions for her opponents, converging to $(b_{j,\star})_{j \neq i}$. Then for all $\lambda > 0$,*

$$\lim_{t \nearrow \infty} u_z^i\left(\left[b_i + \lambda\right] \wedge v^i\left(\cdot; s_i\right), b_{-i,t}; s_i\right) \geq u_z^i\left(b_i, b^{-i,\star}; s_i\right) - \bar{Q}\lambda.$$

Proof. It is sufficient to show that this is the case for any $z \in \operatorname{Supp} Z$. Suppose that $\lim_{t \nearrow \infty} q^i(b_i, b_{-i,t}; z) \geq q^i(b_i, b_{-i,\star}; z)$. Since bids are bounded above by marginal values, this implies that

$$\lim_{t \nearrow \infty} u^i\left(b_i, b_{-i,t}; s_i, z\right) \geq u^i\left(b_i, b_{-i,\star}; s_i, z\right).$$

Condition 3B and straightforward algebra then imply the desired result. Then if the desired result is violated it must be that there is z such that

$$\lim_{t \nearrow \infty} u^i\left(b_i, b_{-i,t}; s_i, z\right) < u^i\left(b_i, b_{-i,\star}; s_i, z\right).$$

By the definition of u^i, it follows that

$$\lim_{t \nearrow \infty} q_{i,t} = \lim_{t \nearrow \infty} q^i\left(b_i, b_{-i,t}; z\right) < q^i\left(b_i, b_{-i,\star}; z\right) = q_{i,\star}.$$

Market clearing implies that there is $J \neq \emptyset$, $i \notin J$, such that for all $j \in J$, $\lim_{t \nearrow \infty} q_{j,t} > q_{j,\star}$. Then for all $k \in J \cup \{i\}$, Lemma 33 implies that

$$b_{k,\star}\left(q'\right) = b_{k,\star}\left(q''\right), \ \forall q', q'' \in I\left(\lim_{t \nearrow \infty} q_{k,t}, q_{k,\star}\right).$$

Let $\overline{b}^{\lambda} = [b_i + \lambda] \wedge v^i(\cdot; s_i)$, and let $\bar{q}_i$ be defined as

$$\bar{q}_i = \sup\left\{q' \in [0, q_{i,\star}] : \; v^i\left(q'; s_i\right) > b_i\left(q'\right)\right\}.$$

Then $\bar{q}_i > \lim_{t\nearrow\infty} q_{i,t}$,[59] and for any $\delta > 0$ there is $\varepsilon > 0$ such that for all $q' \in (\lim_{t\nearrow\infty} q_{i,t}, \bar{q}_i - \delta)$, $\overline{b}^{\lambda}(q') > b_i(q') + \varepsilon$. Since for each $j \in J$, $b_{j,t}$ is converging on $(q_{j,\star}, \lim_{t\nearrow\infty} q_{j,t})$, it follows that for t sufficiently large it must be that

$$q^i\left(\overline{b}^{\lambda}, b_{-i,t}; z\right) \geq \bar{q}_i - \delta.$$

Since δ may be arbitrarily small, it follows that $\lim_{t\nearrow\infty} q^i(\overline{b}^{\lambda}, b_{-i,t}; z) \geq \bar{q}_i$. Then

$$\begin{aligned}
\lim_{t\nearrow\infty} u^i\left(\overline{b}^{\lambda}, b_{-i,t}; z\right) &= \lim_{t\nearrow\infty} \int_0^{q^i\left(\overline{b}^{\lambda}, b_{-i,t}; z\right)} v^i\left(x; s\right) - \overline{b}^{\lambda}\left(x\right) dx \\
&\geq \int_0^{\bar{q}_i} v^i\left(x; s\right) - \overline{b}^{\lambda}\left(x\right) dx \\
&= \int_0^{q^i\left(b_i, b_{-i,\star}; z\right)} v^i\left(x; s\right) - \overline{b}^{\lambda}\left(x\right) dx \\
&\geq u^i\left(b_i, b_{-i,\star}; z\right) - \int_0^{q^i\left(b_i, b_{-i,\star}; z\right)} \overline{b}^{\lambda}\left(x\right) - b_i\left(x\right) dx \\
&\geq u^i\left(b_i, b_{-i,\star}; z\right) - \bar{Q}\lambda.
\end{aligned}$$

Since this holds for all z, the result follows. □

Lemma 35 (Pay-as-bid surplus splitting)**.** *The pay-as-bid auction model satisfies Assumption 5D.*

Proof. Suppose that there is a sequence of strategies $\langle(\beta^{k,t})_{k=1}^n\rangle_{t=1}^\infty$ converging to the feasible strategy profile $(\beta^{\star,k})_{k=1}^n$ such that there is an agent i, a positive-measure set $S_i \subseteq (0,1)$, and for each $s_i \in S_i$ a positive-measure set $S_{-i}(s_i) \subseteq (0,1)^{n-1}$ such that for all $s_i \in S_i$ and $s_{-i} \in S_{-i}(s_i)$,

$$\lim_{t\nearrow\infty} u_z^i\left(\beta^{i,t}\left(s_i\right), \beta^{-i,t}\left(s_{-i}\right); s_i\right) > u_z^i\left(\beta^{i,\star}\left(s_i\right), \beta^{-i,\star}\left(s_{-i}\right); s_i\right).$$

Then for these same $s = (s_i, s_{-i})$,

$$\lim_{t\nearrow\infty} \mathbb{E}_z\left[q^i\left(\beta^t\left(s\right); z\right)\right] > \mathbb{E}_z\left[q^i\left(\beta^\star; z\right)\right].$$

It follows that

$$\Pr_{s,z}\left(\lim_{t\nearrow\infty} q^i\left(\beta^t\left(s\right); z\right) > q^i\left(\beta^\star\left(s\right); z\right)\right) > 0.$$

[59] Otherwise, utility converges.

By market clearing, there is an agent j such that

$$\Pr_{s,z}\left(\lim_{t\nearrow\infty} q^j\left(\beta^t\left(s\right);z\right) < q^j\left(\beta^\star\left(s\right);z\right)\right) > 0.$$

For $\lambda > 0$, consider the deviation $\overline{\beta}_\lambda^{j,t}$ for this agent, given by

$$\overline{\beta}_\lambda^{j,t}\left(s_j\right) = \left[\beta^{j,t}\left(s_i\right) + \lambda\right] \wedge v^j\left(\cdot;s_j\right).$$

Lemma 36 implies that for all s_{-j} and z,

$$u^i\left(\overline{\beta}_\lambda^{j,t}\left(s_j\right), \beta^{-j,t}\left(s_{-j}\right); s_j, z\right) \geq u^j\left(\beta^{j,t}\left(s_j\right), \beta^{-j,t}\left(s_{-j}\right); s_j, z\right) - \bar{Q}\lambda.$$

Then it will suffice to show that the deviation is discretely utility-improving on a set of positive measure, since the above inequality directly implies that

$$u_z^i\left(\overline{\beta}_\lambda^{j,t}\left(s_j\right), \beta^{-j,t}\left(s_{-j}\right); s_j\right) \geq u_z^j\left(\beta^{j,t}\left(s_j\right), \beta^{-j,t}\left(s_{-j}\right); s_j\right) - \bar{Q}\lambda.$$

At this point, if marginal values are strictly decreasing in quantity the proof is essentially complete; additional care must be taken to handle the case in which marginal values are potentially (locally) constant. Lemma 33 establishes that for any $s_i \in S_i$, $\beta^{i,\star}(s_i)$ is constant on intervals on which quantity does not converge, hence $\overline{\varphi}^{i,\star}(s_i)$ is discontinuous at this price. Since $\overline{\varphi}^{i,\star}(s_i)$ is a monotone function on a compact domain, it has at most countably-many discontinuities, so at least one such quantity interval is realized with positive probability. Considering such positive-probability intervals, there is a subset of signals $\hat{S}_i \subseteq S_i$ such that these positive-probability intervals intersect, and it is without loss of generality to assume that this subset has positive measure; otherwise, the interval $[0, \bar{Q}]$ can be covered by uncountably-many disjoint sets of positive measure, a contradiction. Lastly, market clearing implies that agent i's quantity loss is some other agent's quantity gain, and since there are only a finite number of agents it is again without loss to assume that in all cases at least some of the discrete gain goes to agent $j \neq i$. Then let $\hat{S}_i \subseteq S_i$ be a positive-measure set be such that there are $q_{i,\ell}, q_{i,r} \in [0, \bar{Q}]$ with

$$\Pr{}_{s_{-i},z}\Bigg(q^i\left(\beta^{i,\star}\left(s_i\right), \beta^{-i,\star}\left(s_{-i}\right); z\right) \leq q_{i,\ell} < q_{i,r} \\ \leq \lim_{t\nearrow\infty} q^i\left(\beta^{i,t}\left(s_i\right), \beta^{-i,t}\left(s_{-i}\right); z\right)\Bigg) > 0.$$

For $s_i \in \hat{S}_i$, let $\hat{S}_{-i}(s_i)$ be given by

$$\hat{S}_{-i}\left(s_i\right) = \Bigg\{\left(s_{-i}, z\right):\ q^i\left(\beta^{i,\star}\left(s_i\right), \beta^{-i,\star}\left(s_{-i}\right); z\right) \leq\ q_{i,\ell} \\ < q_{i,r} \leq \lim_{t\nearrow\infty} q^i\left(\beta^{i,t}\left(s_i\right), \beta^{-i,t}\left(s_{-i}\right); z\right)\Bigg\}.$$

Lemma 33 implies that $\beta^{i,\star}(s_i)$ is constant on $(q_{i,\ell}, q_{i,r})$ for all $s_i \in \hat{S}_i$, and further that it is equal to the bid placed by any agent who, at the limit, receives agent i's sacrificed quantity. Then if $s_i, s_i' \in \hat{S}_i$ are such that $\beta^{i,\star}(s_i) \neq \beta^{i,\star}(s_i')$ on $(q_{i,\ell}, q_{i,r})$, it must be that $\hat{S}_{-i}(s_i) \cap \hat{S}_{-i}(s_i') = \emptyset$. From this and the fact that $\Pr(\hat{S}_{-i}(\cdot)) > 0$, it follows that there is a market-clearing price p and a positive-measure set $\tilde{S}_i \subseteq \hat{S}_i$ of agent i's signal realizations such that for all $s_i, s_i' \in \tilde{S}_i$ and $q, q' \in (q_{i,\ell}, q_{i,r})$,

$$\left[\beta^{i,\star}\left(s_i\right)\right](q) = p = \left[\beta^{i,\star}\left(s_i'\right)\right]\left(q'\right).$$

Now let S be defined as

$$\begin{aligned} S = \Bigg\{ (s,z) :\ & q^i\left(\beta^{i,\star}\left(s_i\right), \beta^{-i,\star}\left(s_{-i}\right); z\right) \leq\ q_{i,\ell} \\ & < q_{i,r} \leq \lim_{t \nearrow \infty} q^i\left(\beta^{i,t}\left(s_i\right), \beta^{-i,t}\left(s_{-i}\right); z\right), \\ & \text{and } \left[\beta^{i,\star}\left(s_i\right)\right](q) = \left[\beta^{i,\star}\left(s_i\right)\right]\left(q'\right)\ \forall q, q' \in \left(q_{i,\ell}, q_{i,r}\right), \\ & \text{and } \lim_{t \nearrow \infty} q^j\left(\beta^{j,t}\left(s_j\right), \beta^{-j,t}\left(s_{-j}\right); z\right) < q^j\left(\beta^{j,\star}\left(s_j\right), \beta^{-j,\star}\left(s_{-j}\right); z\right) \Bigg\}. \end{aligned}$$

By assumption, there is a positive-probability set $\hat{S}_{-j} \subseteq S$, a positive-measure set $S_j \subseteq \{s_j : (s,z) \in S\}$, and $q_{j,\ell}, q_{j,r} \in [0, \bar{Q}]$ such that for all $(s_{-j}, z) \in S$ and $s_j, s_j' \in S_j$,

$$\begin{aligned} & \lim_{t \nearrow \infty} q^j\left(\beta^{j,t}\left(s_j\right), \beta^{-j,t}\left(s_{-j}\right); z\right) \leq q_{j,\ell} < q_{j,r} \leq q^j\left(\beta^{j,\star}\left(s_j\right), \beta^{-j,\star}\left(s_{-j}\right); z\right) \\ & \qquad \text{and } \left[\beta^{j,\star}\left(s_j\right)\right](q) = p = \left[\beta^{j,\star}\left(s_j'\right)\right]\left(q'\right)\ \forall q, q' \in \left(q_{j,\ell}, q_{j,r}\right). \end{aligned}$$

Since $v^j(q;\cdot)$ is strictly increasing for all q, it is without loss to assume that for all $s_j \in S_j$, $[\beta^{j,\star}(s_j)](q_{j,r}) < v^j(q_{j,r}; s_j)$. Then for λ sufficiently small, $[\overline{\beta}^{\lambda}(s_j)](q_{j,r}) < v^j(q_{j,r}; s_j)$. Then near the limit, $\overline{\beta}^{\lambda}(s_j)$ will yield a discrete increase in utility at cost bounded above by $\bar{Q}\lambda$, and for λ sufficiently small this deviation is discretely profitable on a positive-probability set. Since costs are everywhere bounded by $\bar{Q}\lambda$, Condition 5D is satisfied. □

Lemma 36 (Pay-as-bid constraint super-technicality)**.** *For any agent i and signal s_i, any $y \in \hat{Y}$, and any $\lambda > 0$, define $\overline{b}^{\lambda} = [y + \lambda] \wedge v^i(\cdot; s_i)$. Then $U^i(\overline{b}^{\lambda}, \beta^{-i}; s_i) \geq U^i(y, \beta^{-i}; s_i) - \bar{Q}\lambda$.*

Proof. I show that this result holds ex post, given realized bids for opponents $b_{-i} = \beta^{-i}(s_{-i})$ and model randomness $z = (z_Q, z_q)$. Two cases arise: first, suppose that

$q^i(\overline{b}^\lambda, b_{-i}; z) \geq q^i(y, b_{-i}; z)$. Then since $\overline{b}^\lambda \leq v^i(\cdot; s_i)$ ex-post utility is

$$\begin{aligned} u^i\left(\overline{b}^\lambda, b_{-i}; s_i, z\right) &= \int_0^{q^i\left(b^\lambda, b_{-i}; z\right)} v^i\left(x; s_i\right) - b^\lambda\left(x\right) dx \\ &\geq \int_0^{q^i\left(y, b_{-i}; z\right)} v^i\left(x; s_i\right) - b^\lambda\left(x\right) dx \\ &\geq \int_0^{q^i\left(y, b_{-i}; z\right)} v^i\left(x; s_i\right) - \left[y\left(x\right) + \lambda\right] dx \geq u^i\left(y, b_{-i}; z\right) - \bar{Q}\lambda. \end{aligned}$$

Suppose instead that $q^i(\overline{b}^\lambda, b_{-i}; z) < q^i(y, b_{-i}; z)$; since quantity is increasing in bid, it must be that

$$\lim_{q' \searrow q^i\left(\overline{b}^\lambda, b_{-i}; z\right)} \overline{b}^\lambda\left(q'\right) \leq \lim_{q'' \nearrow q^i\left(y, b_{-i}; z\right)} y\left(q''\right).$$

Monotonicity of values implies that $v^i(q; s_i) \leq y(q)$ for all $q \in (q^i(\overline{b}^\lambda, b_{-i}; z), q^i(y, b_{-i}; z))$. Then

$$\begin{aligned} u^i\left(\overline{b}^\lambda, b_{-i}; s_i, z\right) &= \int_0^{q^i\left(\overline{b}^\lambda, b_{-i}; z\right)} v^i\left(x; s_i\right) - \overline{b}^\lambda\left(x\right) dx \\ &\geq \int_0^{q^i\left(\overline{b}^\lambda, b_{-i}; z\right)} v^i\left(x; s_i\right) - y\left(x\right) dx - \bar{Q}\lambda \\ &\geq \int_0^{q^i\left(y, b_{-i}; z\right)} v^i\left(x; s_i\right) - y\left(x\right) dx - \bar{Q}\lambda = u^i\left(y, b_{-i}; s_i, z\right) - \bar{Q}\lambda. \end{aligned}$$

Then regardless of the bids b_{-i} submitted by agent i's opponents,

$$u^i\left(b^\lambda, b_{-i}; s_i, z\right) \geq u^i\left(y, b_{-i}; s_i, z\right) - \bar{Q}\lambda.$$

□

Corollary 5 (Probabilistic convergence of observables)**.** *Let $q : \hat{Y} \times Supp\ Z \to \mathbb{R}^n$ and $\pi : \hat{Y} \times Supp\ Z \to \mathbb{R}$ represent allocations and revenue, respectively, in the divisible-good model $\mathcal{M}$. If $\langle(\beta^t)\rangle_{t=1}^\infty$ is a sequence of monotone pure-strategy equilibria in the ε^t-discretized models converging to the supremum-limit strategy profile $(\overline{\beta})$, then*

$$q\left(\beta^t\left(s\right); z\right) \xrightarrow{P} q\left(\overline{\beta}\left(s\right); z\right), \text{ and } \pi\left(\beta^t\left(s\right); z\right) \xrightarrow{P} \pi\left(\overline{\beta}\left(s\right); z\right).$$

Proof of Corollary 5. That quantity is utility-relevant follows from the logic employed in the proof of Lemma 35, which establishes Condition 5D. In particular, if quantity is not converging for a positive-measure set of signal realizations, it is without loss to assume that agent i loses quantity in the limit. Then for some realizations of agent signals, the lost quantity intervals overlap, and Lemma 33 then implies that for some of these type realizations the lost quantity intervals overlap at exactly the same bid level. Then because marginal values are strictly monotone in signal, a positive-measure subset of these signal realizations is such that agent i's utility is not converging, implying that quantity is utility-relevant.

With regard to seller revenue, that bidding strategies are converging implies that if $\pi(y^\star(s), z) \neq \lim_{t\nearrow\infty} \pi(y^t(s); z)$, then $q(y^\star(s); z) \neq \lim_{t\nearrow\infty} q(y^t(s); z)$. Then since q is utility-relevant, π is utility-relevant. □

D Proofs for subsection 4.1: equilibrium approximation

Theorem 4 (Equilibrium approximation)**.** *Let (W, T_W) be a topological space, and suppose that $o : \hat{Y}^n \times Supp\ Z \to W$ is utility-relevant. Then for almost all $s \in (0,1)^n$,*

$$\lim_{t\nearrow\infty} o\left(\alpha^t(s); z\right) = o\left(\overline{\alpha}(s); z\right).$$

Proof. This is an immediate consequence of utility-relevance of o and the construction of $\overline{\alpha}^i$ as a limit of $\alpha^{i,t}$ at which almost all utilities converge. □

Theorem 5 (Probabilistic approximation of observables)**.** *Let $o : \hat{Y}^n \times SuppZ \to \mathbb{R}^m$ be utility-relevant. Then*

$$o\left(\alpha^t(s); z\right) \xrightarrow{P} o\left(\overline{\alpha}(s); z\right).$$

Proof. To begin, fix z and s. For any $\lambda > 0$, utility relevance implies that there is T such that $|o(\alpha^t(s); z) - o(\overline{\alpha}(s); z)| < \lambda$ for all $t > T$. Now fix only z, and suppose that

$$\lim_{\lambda'\searrow 0} \lim_{t\nearrow\infty} \Pr_s\left(\left|o\left(\alpha^t(s); z\right) - o\left(\overline{\alpha}(s); z\right)\right| \geq \lambda'\right) > 2\delta > 0.$$

Then there is $\lambda' > 0$ and a positive-measure set S such that

$$\lim_{t\nearrow\infty} \left|o\left(\alpha^t(s); z\right) - o\left(\overline{\alpha}(s); z\right)\right| > \delta.$$

Then for all $s \in S$, there is no T such that $|o(\alpha^t(s); z) - o(\overline{\alpha}(s); z)| < \delta$ for all $t \geq T$, contradicting the previous argument. Then $o(\alpha^t(s); z) \xrightarrow{P|z} o(\overline{\alpha}(s); z)$. Letting z be free, the same arguments imply the desired result. □