

*These notes are rough; this is mostly in order to get them out before the homework is due. If you would like things polished/clarified, please let me know.*

## Expenditure minimization

Until this point we have been considering the problem of a consumer who would like maximize her utility given a particular budget constraint. There is a related problem which takes the opposite view: given a level of utility and market prices, what is the least amount of wealth necessary to achieve this level of utility? This problem is referred to as the *expenditure minimization problem*.

Although not as direct as the utility maximization problem, you can consider the expenditure minimization problem in this way: given the level of utility to be obtained from consumption of a particular set of goods, what is the least you have to spend? By minimizing this expenditure, you maximize the spending power left over for purchase of other goods. A possible analogy may be in car shopping: first you decide the make and model that you want, then you price-compare across dealerships to find the best terms.

Given that utility is arbitrary — that is, the absolute number of utility does not matter, only its comparison to other numbers — this approach is a little ad hoc and is certainly less well-motivated than utility maximization. In the next section (not the following subsection) we will address the real value of solving the expenditure minimization problem.

## Hicksian demand

When we find an agent's optimal consumption in terms of wealth and prices, we have solved for the *Marshallian demand function*  $x(p, w)$ ; this terminology has not been used previously, since there has been no ambiguity in the nature of the question. The solution to the expenditure minimization problem, on the other hand, is referred to as *Hicksian demand* and is denoted  $h(p, u)$ . In particular, Hicksian demand in an economy with  $\ell$  commodities is given by

$$h(p, \mathbf{u}) = \operatorname{argmin}_x \sum_{i=1}^{\ell} p_i x_i \text{ s.t. } u(x) \geq \mathbf{u}.$$

Where before we were finding the argmax (the consumption bundle which maximizes utility) we are now finding the argmin (the consumption bundle which minimizes total cost).

In the context of Econ 11, an argument similar to Walras' Law (see handout 2) implies that  $u(x) = \mathbf{u}$ : since utility is strictly monotonic and continuous, if  $u(x) > \mathbf{u}$  then we can reduce consumption slightly without slipping below the utility constraint; since this reduces the final cost of consumption, it follows that  $x$  cannot minimize expenditure. Hicksian demand, then, can be viewed as the solution to

$$h(p, \mathbf{u}) = \operatorname{argmin}_x \sum_{i=1}^{\ell} p_i x_i \text{ s.t. } u(x) = \mathbf{u}.$$

Now, as with the utility maximization problem there are several methods available for solving this constrained minimization problem. However, the most direct method directly follows our intuitive argument for utility maximization (see handout 2). In particular, we still must have that the indifference curve through the optimal bundle should be tangent to the budget frontier, hence our solution should still solve

$$\frac{\text{MU}_{x_i}}{p_i} = \frac{\text{MU}_{x_j}}{p_j}$$

for all  $i$  and  $j$ . Once we have found the system which solves the marginal utility per unit cost equations, we substitute into the *utility* constraint to find the cost-minimizing bundle. In the Marshallian case, we

substitute into the budget constraint, but since in the Hicksian case it is utility which is constrained, we use the utility constraint instead. This is clarified in the following example.

### Example: Cobb-Douglas utility

**Question:** suppose that there are two commodities,  $x$  and  $y$ , and that  $u(x, y) = x^{\frac{1}{3}}y^{\frac{2}{3}}$ . Prices are  $p_x = 2$ ,  $p_y = 1$ . What is the expenditure-minimizing bundle if the agent would like to obtain a utility of  $\mathbf{u} = 4$ ?

**Solution:** we know that Hicksian demand  $h((2, 1), 4)$  will be found through solving the marginal utility per unit cost equations. In particular, we have

$$\begin{aligned} \text{MU}_x &= \frac{1}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}}, \\ \text{MU}_y &= \frac{2}{3}x^{\frac{1}{3}}y^{-\frac{1}{3}}. \end{aligned}$$

We may then check

$$\begin{aligned} \iff & \frac{\text{MU}_x}{p_x} = \frac{\text{MU}_y}{p_y} \\ \iff & \frac{\frac{1}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}}}{2} = \frac{\frac{2}{3}x^{\frac{1}{3}}y^{-\frac{1}{3}}}{1} \\ & y = 4x. \end{aligned}$$

Substituting into the utility constraint, we have

$$x^{\frac{1}{3}}y^{\frac{2}{3}} = 4 \implies x^{\frac{1}{3}}(4x)^{\frac{2}{3}} = 4 \implies x^* = 4^{\frac{1}{3}}.$$

Returning to our equation for  $y$  in terms of  $x$ , we find

$$y^* = 4x^* = 4\left(4^{\frac{1}{3}}\right) = 4^{\frac{4}{3}}.$$

The solution to the expenditure minimization problem is then given by

$$h((2, 1), 4) = (x^*, y^*) = \left(4^{\frac{1}{3}}, 4^{\frac{4}{3}}\right).$$

□

### Corner solutions

This is mainly a note of caution: as we have already discussed corner solutions (see handout 2), there is no use in belaboring the point further. If you solve an expenditure minimization problem and obtain negative consumption, you will need to adjust your answer appropriately. In particular, rather than finding the intersection of the budget frontier with the axis, you will need to find the intersection of the *indifference curve* with the appropriate axis. As before, so long as the Inada conditions are satisfied you will not need to worry about corner solutions, so in the case of Cobb-Douglas and CES utilities there is no issue; however, when utility is (say) quasilinear you will need to check for this potential problem.

### Welfare effects of price changes

It is clear that when prices change, consumption changes. One key desire of economics is to describe the nature of these changes; while we can find Marshallian demand  $x(p, w)$  at current prices  $p$  as well as at new

prices  $p'$  to express explicitly how consumption has changed, the standard solution to the optimal demand provides no insight into precisely *why* consumption shifts in this particular way.

To make this more concrete, suppose that we start at a price vector  $p$ , then move to a new price vector  $p'$  which is identical to  $p$  except that the price of good  $i$  has decreased. Provided the economy isn't too unusual, we should expect that:

- Consumption of good  $i$  will increase. As its price drops it becomes more affordable, so we will buy more.
- Consumption of goods  $j \neq i$  will decrease. This is a corollary to the above point. When we invest more in good  $i$ , we sacrifice some consumption of all (or most) other goods. This translation from goods  $j$  to good  $i$  is referred to as a *substitution effect*.
- Consumption of goods  $j \neq i$  will increase; this is an intentional contradiction of the above point. When the price of good  $i$  falls, we can afford our old level of consumption and still have wealth to spare; this leftover wealth may be invested in other goods. This appearance of purchasing power "from nowhere" is referred to as a *wealth effect*.

Of course, each of these points holds in the opposite direction if instead the price of good  $i$  increases.

## Quantifying the effects

We consider wealth and substitution effects as arising from the shift of prices from  $p$  to  $p'$ ; we would like to account for the difference in consumption

$$x(p', w) - x(p, w).$$

Mathematically, we can see that for any  $c$  this expression is identical to

$$x(p', w) - x(p, w) = x(p', w) - x(p, w) + c - c = [x(p', w) - c] + [c - x(p, w)].$$

Expressing wealth and substitution effects amounts to selecting the proper  $c$  in this context.

If optimal demand at prices  $p$  is given by  $x(p, w)$ , the agent obtains utility from this level of consumption of  $u(x(p, w))$ ; traditionally, this is defined to be  $v(p, w) \equiv u(x(p, w))$ , where  $v(\cdot, \cdot)$  is called the *indirect utility function*.  $v(\cdot, \cdot)$  maps prices and wealth directly to utility, without immediate consideration of what consumption leads to this utility.

In light of Hicksian demand, we define  $c$  by

$$c = h(p', v(p, w));$$

that is,  $c$  is expenditure-minimizing demand at prices  $p'$ , sufficient to reach the utility the agent previously obtained at prices  $p$ ,  $v(p, w)$ .

We can then view the change in consumption as

$$x(p', w) - x(p, w) = [x(p', w) - h(p', v(p, w))] + [h(p', v(p, w)) - x(p, w)].$$

The left-hand term is the wealth effect and the right-hand term is the substitution effect; in this view, overall consumption changes are merely the sum of the wealth and substitution effects.

Since remembering which effect is which can be difficult, here is a simple way of keeping them straight:

- The wealth effect  $x(p', w) - h(p', v(p, w))$  is the term which holds prices fixed. In this term, the only parameter changing is wealth, so this tracks how changes to purchasing power alter consumption.

- The substitution effect  $h(p', v(p, w)) - x(p, w)$  is the term which is not the wealth effect. This term tracks how optimal consumption along the indifference curve defined by  $\bar{u} = v(p, w)$  changes with prices.

### Example: Cobb-Douglas utility

**Question:** return to our Cobb-Douglas example from above, where  $u(x, y) = x^{\frac{1}{3}}y^{\frac{2}{3}}$ , and suppose that the agent has wealth  $w = 12$ . Suppose that the initial price is  $p = (1, 2)$  and the new price is  $p' = (2, 1)$ . Find the wealth and substitution effects.

**Solution:** to begin, we need to find  $x(p, w)$  and  $x(p', w)$ . We can do this using the usual marginal utility per unit cost equation, but with Cobb-Douglas we have a shortcut: expenditure on a particular good is proportional to its exponent. In particular,

$$x^* = \left(\frac{1}{3}\right) \frac{w}{p_x}, \quad y^* = \left(\frac{2}{3}\right) \frac{w}{p_y}.$$

With  $w = 12$ , we can then see

$$x(p, 12) = (4, 4), \quad x(p', w) = (2, 8).$$

To find utility  $v(p, w)$ , we take the optimal consumption at  $(p, w)$  and substitute it into the utility function,

$$v(p, w) = u(x(p, w)) = u(4, 4) = 4^{\frac{1}{3}}4^{\frac{2}{3}} = 4.$$

The last step is to obtain Hicksian demand  $h(p', v(p, w)) = h(p', 4)$ ; however, in the previous example we have already found that

$$h(p', 4) = h((2, 1), 4) = \left(4^{\frac{1}{3}}, 4^{\frac{4}{3}}\right).$$

The wealth and substitution effects are then given by

$$\begin{aligned} x(p', w) - h(p', v(p, w)) &= (2, 8) - \left(4^{\frac{1}{3}}, 4^{\frac{4}{3}}\right) \\ &= \left(2 - 4^{\frac{1}{3}}, 8 - 4^{\frac{4}{3}}\right) \\ &\approx (0.413, 1.650); \\ h(p', v(p, w)) - x(p, w) &= \left(4^{\frac{1}{3}}, 4^{\frac{4}{3}}\right) - (4, 4) \\ &= \left(4^{\frac{1}{3}} - 4, 4^{\frac{4}{3}} - 4\right) \\ &\approx (-2.413, 2.350). \end{aligned}$$

Please note that the decimal approximations are not important, and serve only to check the signs of the wealth and substitution effects.

These effects are demonstrated graphically in Figure 1. □

## Production: the firm's problem

*This section will be filled out following Thursday's lecture.*

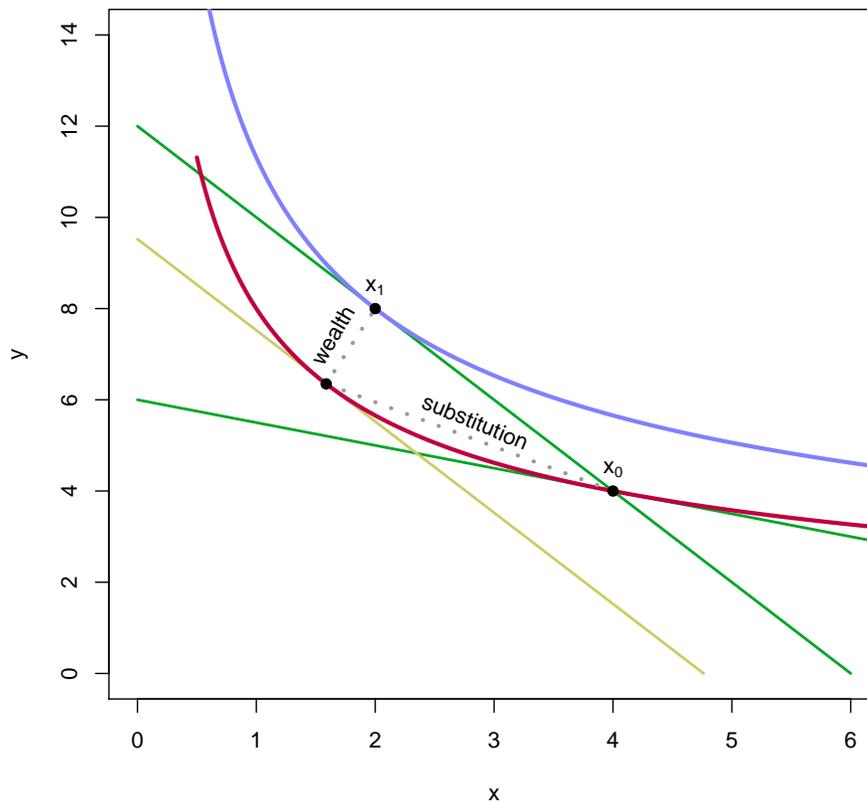


Figure 1: welfare effects in the Cobb-Douglas example;  $x_0$  is original optimal consumption, and  $x_1$  is new optimal consumption. While overall utility is improved when prices shift from  $(1, 2)$  to  $(2, 1)$ , we can decompose the shift into wealth and substitution effects. The substitution effect slides consumption along the same indifference curve to a point which is tangent to the new price vector; the wealth effect shifts consumption out from this point to return wealth to its original value. Note that the “rotation” of the budget frontier through  $x_0$  is a consequence of the particular wealth and prices used in this problem, and is not a general feature.