*This document papers over some important theoretical and some kind-of-important technical considerations, in the interest of obtaining results without getting bogged down in details. It is worth remembering that Econ 106D is an undergraduate course, and it is the economic intuition — which can be followed through simple math — which is important; robustness comes later.*

## An all-pay auction

Consider a high-bid-wins auction format in which each bidder is forced to pay her bid, irrespective of whether or not she won the item. This is not a friendly way to run an auction! Although it may seem intuitive that this is a good way to separate bidders from their money, we will see that this format encourages bidders to bid well below their true values, mitigating the effect on the seller's expected revenue.[1]

Suppose that there are $N$ bidders $i \in \{1, \ldots, N\}$, each with valuation $v_i \sim U(0,1)$. Bidders submit bids $b_i$ to the seller, and the bidder with the highest bid wins the item (thereby obtaining utility $v_i - b_i$); all bidders pay their bids, regardless of whether or not their bids are highest.

We can see that a bidder's utility maximization problem is expressed as

$$\max_b \Pr\left(b > b_j \forall j \neq i\right) v_i - b = \max_b w\left(b\right) v_i - b.$$

That is, the bidder obtains the item — and hence her value $v_i$ — with probability equal to the probability that she wins the item, which is the probability that her bid is highest; she pays her bid $(-b)$ regardless. We use the function $w(\cdot)$ to represent the probability that she wins the item, to avoid complications writing $\Pr(\cdot)$ and its derivatives.

First-order conditions tell us:

$$\frac{\partial}{\partial b} : w'(b)v_i - 1 = 0 \qquad \Longrightarrow \qquad w'(b)v_i = 1.$$

We make the conjecture that, in equilibrium, agents bid $b(v) = \alpha v^N$. Letting $i = 1$ to keep our notation simple, this gives us

$$\begin{aligned}
w(b) &= \Pr\left(b > b_j \forall j \neq i\right) \\
&= \Pr\left(b > b_2, b > b_3, \ldots, b > b_N\right) \\
&= \Pr\left(b > b_2\right) \times \Pr\left(b > b_3\right) \times \cdots \times \Pr\left(b > b_N\right) \\
&= \prod_{j=2}^{N} \Pr\left(b > b_j\right) \\
&= \prod_{j=2}^{N} \Pr\left(b > \alpha v_j^N\right) \\
&= \prod_{j=2}^{N} \Pr\left(v_j < \sqrt[N]{\frac{b}{\alpha}}\right) \\
&= \prod_{j=2}^{N} \Pr\left(v_j \leq \sqrt[N]{\frac{b}{\alpha}}\right) \\
&= \prod_{j=2}^{N} \sqrt[N]{\frac{b}{\alpha}} \\
w(b) &= \left(\frac{b}{\alpha}\right)^{\frac{N-1}{N}}.
\end{aligned}$$

[1] After all, revenue equivalence tells us that this format cannot result in strictly greater *expected* revenue for the seller.

It follows that

$$w'(b) = \left(\frac{N-1}{N\alpha}\right)\left(\frac{b}{\alpha}\right)^{-\frac{1}{N}}.$$

Returning to the expression we found from first-order conditions, we substitute in the expression for $w'(\cdot)$ and find

$$\begin{aligned} & w'(b)v_i = 1 \\ \iff \quad & \left(\frac{N-1}{N\alpha}\right)\left(\frac{b}{\alpha}\right)^{-\frac{1}{N}} v_i = 1 \\ \iff \quad & \left(\frac{N-1}{N\alpha}\right) v_i = \left(\frac{b}{\alpha}\right)^{\frac{1}{N}} \\ \iff \quad & \left(\frac{N-1}{N\alpha}\right)^N v_i^N = \frac{b}{\alpha} \\ \iff \quad & \left(\frac{N-1}{N}\right)^N v_i^N \alpha^{1-N} = b. \end{aligned}$$

Recall that we conjectured $b(v) = \alpha v^N$. We can use this conjecture and the above expression to solve for $\alpha$,

$$\alpha v^N = \left(\frac{N-1}{N}\right)^N v^N \alpha^{1-N} \quad \Longrightarrow \quad \alpha^N = \left(\frac{N-1}{N}\right)^N \quad \Longrightarrow \quad \alpha = \frac{N-1}{N}.$$

Putting all of this together, we find

$$b(v_i) = \left(\frac{N-1}{N}\right) v_i^N.$$

This constitutes a Nash equilibrium in bidding strategies. Since we know that equilibrium in the first-price auction with the same bidder setup entails bidding strategies of $b_{\text{FP}}(v) = (N-1)/N \times v$, we can see that bids in the all-pay setup are *significantly* lower: with $v_i \in (0,1)$, we have $v_i^N < v_i$. Thus requiring all bidders to pay their bids ensures that bidders won't bid very much for the item.

## Revenue equivalence

One of the central results of auction theory is known as *revenue equivalence*: any two auction formats which have efficient outcomes and don't require a bidder with the lowest possible valuation to pay anything must result in the same expected revenue to the seller.[2][3] Although this may elicit a yawn and a solid "BFD," the result is itself pretty incredible; to a first approximation, it says that the rules of the game don't matter! As long as the person who values the item most highly gets it, what we make them do for it is irrelevant.

In parallel, it is fairly intuitive that the expected highest valuation is independent of the rules of the auction: a bidder's value does not depend on the auction being run. This means that *expected total surplus is fixed,* so long as we are guaranteed that the bidder with the highest valuation obtains the item. Since surplus is split between the buyers and the seller — all the pie must go to one side or the other — revenue has the additional implication of *expected utility equivalence*: a bidder's assessment of her expected utility is independent of the auction format.[4]

This feature gives us a new method of finding equilibrium bidding strategies. We will see this in a moment.

[2] For this result many economists thank Myerson (1981). However, it was independently discovered and published roughly concurrently by UCLA's own John Riley, who may have taught your 106P course. Academic credit is a harsh mistress.

[3] There are, as in many situations in this class, some additional technical considerations; for the purposes of 106D these are extraneous.

[4] Again, technical considerations, but our approximation is good enough.

## An example

Consider a second-price auction with $N = 2$ bidders, each with valuation $v_i \sim F$, where $F$ is a CDF given by

$$F(v) = v^2, \quad v \in [0, 1].$$

Note that this implies that the density $f$ is given by $f(v) = 2v$.

We have justified that in the second-price auction bidders will bid their valuations; this result is independent of the underlying distribution of valuations. We can therefore see that a bidder's equilibrium expected utility is equal to (using some hand-wavy notation)

$$\mathbb{E}_{\text{SP}}[u_i] = \Pr(v_i \text{ is highest})(v_i - \mathbb{E}[\text{second-highest } v_j | v_i \text{ is highest}]).$$

That is, with probability equal to the probability that her value is the highest, bidder $i$ receives her valuation $v_i$ less the expected second-highest bid — which, in equilibrium, is the second-highest value. The payment is in expectation, since we are dealing with expected utility.

With two bidders, the expression $\mathbb{E}[\text{second-highest } v_j | v_i \text{ is highest}]$ is equivalent to $\mathbb{E}[v_j | v_i > v_j]$, since knowing that $i$'s valuation is the highest implies that the other bidder's valuation is the second-highest (lowest)! We know that such an expectation can be computed as

$$\mathbb{E}[v_j | v_i > v_j] = \int_0^{v_i} v_j f(v_j | v_i > v_j)\, dv_j.$$

This is now a matter of computing the conditional PDF. By definition, we have

$$f(v_j | v_i > v_j) = \frac{f(v_j)}{\Pr(v_i > v_j)} = \frac{f(v_j)}{\Pr(v_j \leq v_i)} = \frac{f(v_j)}{F(v_i)} = \frac{2v_j}{v_i^2}.$$

Working through the expectation, we have

$$\begin{aligned}
\mathbb{E}[v_j | v_i > v_j] &= \int_0^{v_i} v_j \left(\frac{2v_j}{v_i^2}\right) dv_j \\
&= \left(\frac{1}{v_i^2}\right) \int_0^{v_i} 2v_j^2 dv_j \\
&= \left(\frac{1}{v_i^2}\right)\left(\frac{2}{3}\right) v_j^3 \Big|_{v_j=0}^{v_i} \\
&= \frac{2}{3} v_i.
\end{aligned}$$

It follows that $i$'s expected utility is

$$\mathbb{E}_{\text{SP}}[u_i] = \Pr(v_i \text{ is highest})\left(v_i - \frac{2}{3}v_i\right) = \frac{1}{3}\Pr(v_i \text{ is highest})\, v_i.$$

We now ask the question, what should one of these bidders do in the related first-price auction? In the first-price auction, we can express expected utility as

$$\mathbb{E}_{\text{FP}}[u_i] = \Pr(b_i \text{ is highest})(v_i - b_i).$$

In the natural equilibrium of a symmetric first-price auction, we know that the outcome will be efficient; thus bidder $i$ only wins — has the highest bid — when her valuation is the highest. It follows that the expression $\Pr(b_i \text{ is highest})$ can be replaced with $\Pr(v_i \text{ is highest})$. Thus we have

$$\mathbb{E}_{\text{FP}}[u_i] = \Pr(v_i \text{ is highest})(v_i - b_i).$$

Expected utility equivalence implies that $\mathbb{E}_{\text{SP}}[u_i] = \mathbb{E}_{\text{FP}}[u_i]$. Then we have

$$\frac{1}{3}\Pr\left(v_i \text{ is highest}\right) v_i = \Pr\left(v_i \text{ is highest}\right)\left(v_i - b_i\right) \quad \Longrightarrow \quad b_i = \frac{2}{3}v_i.$$

We can use revenue equivalence to compute equilibrium bidding strategies without taking first-order conditions!

### Verifying equilibrium

Having determined by revenue equivalence that $b(v) = (2/3)v$, we check that this is a Nash equilibrium. As we have seen in section, the maximization problem for a participant in a first-price auction is

$$\max_b \Pr(b \text{ is highest})\left(v_i - b\right) = \max_b w(b)\left(v_i - b\right).$$

First-order conditions give us

$$\frac{\partial}{\partial b} : w'(b)\left(v_i - b\right) - w(b) = 0.$$

We want to check that $b(v) = (2/3)v$ is a Nash equilibrium, thus the first-order conditions become

$$w'(b)\left(v_i - \frac{2}{3}v_i\right) = w(b) \quad \Longrightarrow \quad w'(b)v_i = 3w(b).$$

We can also see that

$$\begin{aligned} w(b) &= \Pr\left(b > b_{-i}\right) \\ &= \Pr\left(b > \frac{2}{3}v_{-i}\right) \\ &= \Pr\left(v_{-i} < \frac{3}{2}b\right) \\ &= \Pr\left(v_{-i} \leq \frac{3}{2}b\right) \\ &= F\left(\frac{3}{2}b\right) \\ w(b) &= \left(\frac{9}{4}\right)b^2 \quad \Longrightarrow \quad w'(b) = \left(\frac{9}{2}\right)b \end{aligned}$$

Substituting in to the above equation, this gives

$$\left(\frac{9}{2}\right)bv_i = 3\left(\frac{9}{4}\right)b^2 \quad \Longrightarrow \quad 2v_i = 3b \quad \Longrightarrow \quad b = \frac{2}{3}v_i.$$

Thus our determined equilibrium is confirmed.

## Expected revenue

Lastly, it is occasionally useful to compute the expected revenue of an auction. Recalling revenue equivalence, this can be achieved by computing expected revenue in a second-price auction (so long as the antecedents of revenue equivalence are satisfied). Further, since we have already established that is a weakly-dominant strategy to bid your own valuation in a second-price auction, computing expected revenue in a second-price auction is equivalent to computing the expected second-highest value. Excellent!

Unfortunately, the distribution of the second-highest value is less than obvious. For this, we turn to the notion of an *order statistic.*

## Order statistics

Consider $N$ independent, identically-distributed continuous random variables $\{X_1, \ldots, X_N\}$, with each $X_i \sim F$ for some CDF $F$ (with associated PDF $f$). We are accustomed to speaking of the distribution of a single one of these random variables, or perhaps of the joint distribution of all of the variables together. However, there are random variables which we can derive from these random variables: for example, we can consider the smallest of the bunch. Unlike any particular $X_i$, the distribution of the smallest of the $\{X_1, \ldots, X_N\}$ will not equal $F$: it will tend to be smaller than this distribution would suggest! To make it clear, this is not the distribution of a single random variable which is *maybe* the smallest, it is the distribution of the smallest value, the "$i$" of which will vary depending on the realized values of the random variables.

The $k^{\text{th}}$-smallest value is referred to as the $k^{\text{th}}$ *order statistic* (i.e., "smallest" is $k = 1$, "largest" is $k = N$, etc.); we have a density for the distribution of its value,

$$f_{(k)}(x) = k \begin{pmatrix} N \\ N-k \end{pmatrix} (1 - F(x))^{N-k} F(x)^{k-1} f(x).$$

Although this formula is undeniably ugly, it actually makes a certain amount of sense:

- ${}_N C_{N-k}$: since we are interested in the $k^{\text{th}}$-smallest value, we need $N - k$ values to be larger; ${}_N C_{N-k}$ is the number of ways of choosing these $N - k$ values.
- $(1 - F(x))^{N-k}$: $F(x)$ is the probability that any particular value is *below* $x$, and $1 - F(x)$ is the probability that any particular value is *above* $x$. By independence, the probability that $N - k$ values are above $x$ is $(1 - F(x))^{N-k}$.
- $F(x)^{k-1}$: since we are interested in the $k^{\text{th}}$-smallest value, we need $k - 1$ values to be smaller; since $F(x)$ is the probability that any particular value is below $x$, independence tells us that $F(x)^{k-1}$ is the probability that $k - 1$ values are below $x$.
- $k$: having chosen $N - k$ values to be larger, there are $k$ ways to choose the $k^{\text{th}}$-smallest value.
- $f(x)$: this is the unconditional density associated with value $x$.

*Note*

Under the auspices of revenue equivalence, most expected-revenue calculations reduce to computing the expected second-highest value (as discussed above). However, even if revenue equivalence does not hold the equation for the $k^{\text{th}}$ order statistic can still be informative; we simply cannot apply it as blindly.

## Continuing the previous example

To compute expected revenue for the case where $N = 2$ and $v_i \sim F$, with $F(v) = v^2$, we compute the expected second-highest value (and hence the expected second-highest bid) as the expected first-smallest value. To do so will require knowing the distribution of this value, which we obtain as the distribution of the first order statistic,

$$f_{(1)}(x) = (1) \begin{pmatrix} 2 \\ 1 \end{pmatrix} (1 - F(x))^1 F(x)^0 f(x) = 4(1 - x^2)x.$$

Computing the expectation is a simple matter of taking an integral,

$$\begin{aligned}\mathbb{E}\left[\text{second-highest } v_i\right] &= \int_0^1 v f_{(1)}(v) dv \\ &= \int_0^1 4v^2 - 4v^4 dv \\ &= \frac{4}{3}v^3 - \frac{4}{5}v^5 \Big|_{v=0}^1 = \frac{8}{15}.\end{aligned}$$

For more practice and to "verify" revenue equivalence, we can also use this equation to compute expected revenue for the first-price auction. In this case, we are interested in the expected highest bid,

$$\mathbb{E}\left[\text{highest } b_i\right] = \mathbb{E}\left[\text{highest } \frac{2}{3}v_i\right] = \mathbb{E}\left[\text{second-smallest } \frac{2}{3}v_i\right].$$

The former equality follows from the equilibrium bid functions, $b(v) = (2/3)v$, and the latter follows from simple logic (with two bidders, the highest value is the second-smallest value). The order statistic equation give the density of the second-smallest value as

$$f_{(2)}(x) = (2)\left(\begin{array}{c}2\\0\end{array}\right)(1-F(x))^0 F(x)^1(x) = 4x^3.$$

Computing expected revenue is again a simple matter of taking an integral,

$$\begin{aligned}\mathbb{E}\left[\text{second-smallest } \frac{2}{3}v_i\right] &= \int_0^1 \frac{2}{3} v f_{(2)}(v) dv \\ &= \int_0^1 \frac{8}{3}v^4 dv \\ &= \frac{8}{15}v^5 \Big|_{v=0}^1 = \frac{8}{15}.\end{aligned}$$

The expected revenues are equal in the two auction formats! This is a relief, since it means that economics is on very solid footing.[5]

## Generic expected revenue for the uniform distribution

Consider an efficient auction for an item, where there are $N$ bidders $i \in \{1, \ldots, N\}$ with $v_i \sim U(0,1)$. What is expected revenue?

As before, expected revenue is equal to the expected second-highest value. In turn, this is the expected $N-1^{\text{th}}$-smallest value; the order statistic equation gives its density as

$$f_{(N-1)}(x) = (N-1)\left(\begin{array}{c}N\\1\end{array}\right)(1-F(x))^1 F(x)^{N-2} f(x) = N(N-1)(1-x)x^{N-2}.$$

[5]At this point, it behooves me to suggest that you invest some of your remaining time at UCLA in a logic or philosophy of science course.

We can compute

$$\begin{aligned}\mathbb{E}\left[\text{second-highest } v_i\right] &= \int_0^1 v\left[N(N-1)(1-v)v^{N-2}\right]dv \\ &= N(N-1)\int_0^1 v^{N-1} - v^N dv \\ &= (N-1)v^N - \left(\frac{N(N-1)}{N+1}\right)v^{N+1}\Big|_{v=0}^1 \\ &= (N-1) - \frac{N(N-1)}{N+1} = \frac{N-1}{N+1}.\end{aligned}$$

This is exactly as was suggested (intuitively) in lecture: it is equal to the expected highest value — $N/(N+1)$ — minus the expected bidder surplus — $1/(N+1)$, equal to the expected distance between bids.