Production

As we move away from solving simple consumer problems, we enter the somewhat-more-realistic world of general equilibrium. While introducing a price mechanism to facilitate trade between individuals is useful on some fundamental level, in describing how an economy works it is necessary to add the ability to introduce new products and goods to the system. Enter the firm; in theory, an economic firm represents (more or less) a technology for transforming goods of one type to goods of another. Its ability to produce is given by the *production set*, denote Y. In class, we have discussed various features of the production set; below, we will attack the particular assumption of free disposal, as well as solve two examples of profit-maximizing behavior.

Free disposal

In lecture, there was some discussion regarding the utility behind the free disposal assumption[1]. It seems that the best answer to the question is twofold: (i) firms can actually engage in free (or nearly-free) disposal, at least to the extent that it matters; and (ii) introducing free disposal does not affect profit-maximizing behavior on the part of a price-taking firm. We will discuss each of these points below, in reverse order.

Theorem

Let $Y \subset \mathbb{R}^L$ be a production set, and let $Y' \equiv Y - \mathbb{R}^L_+$ be "Y with free disposal." Then if $p \gg 0$ and $y' \in \operatorname{argmax}_{z \in Y'} p \cdot z$, $y' \in \operatorname{argmax}_{z \in Y} p \cdot z$.

Proof. note that any $z \in Y'$ is such that $z = y_z - r_z$ for some $y_z \in Y$ and $r_z \in \mathbb{R}^L_+$. If y' solves the firm's problem, then we know that $p \cdot y' \geq p \cdot z$ for all $z \in Y'$; in particular, $p \cdot (y_{y'} - r_{y'}) \geq p \cdot z$. Suppose that $r_{y'} \neq 0$. Then $p \cdot y_{y'} > p \cdot y'$, and $y_{y'} = y_{y'} - 0 \in Y'$. Hence $y_{y'}$ is feasible in Y' and provides strictly greater profits than y', a contradiction of optimality. Thus if y' is a maximizer, it must be that $r_{y'} = 0$, which implies $y' \in Y$.

Now suppose that $y' \notin \operatorname{argmax}_{z \in Y} p \cdot z$; then there is some $y'' \in Y$ such that $p \cdot y'' > p \cdot y'$. But since $y'' = y'' - 0 \in Y'$, this contradicts the optimality of y' in Y'. Hence y' also solves the initial profit-maximization problem in the production set Y. □

The purpose of this theorem is to demonstrate just how innocuous the free disposal assumption is in the case of price-taking equilibrium. In particular, if we introduce free disposal to a particular firm's problem, the solution to the problem does not change. Notice, however, that this requires that the firm cannot price strategically.

Ponies and rainbows

As an example of why free disposal might matter with strategic pricing, consider a firm which can produce ponies from rainbows, with a production set given by

$$Y = \{(0,0), (-1,2)\}.$$

That is, the firm can either shut down, or can produce two ponies from one rainbow. There is a single consumer with quasilinear demand for ponies and cold, hard cash, given by

$$u(h,m) = 1[h \geq 1]k + m, \quad k > 1.$$

[1] In section, we discussed the reality of the assumption using a famous (in certain circles) story about Atari, the videogame *ET*, and New Mexico. I highly suggest Googling this little bit of history.

Then if the firm shuts down, the price of a pony must be greater than k; assuming that the price of a rainbow is less than k, the firm would rather produce $(-1, 2)$. However, if the firm produces 2 ponies, a strict market clearing equality (demand precisely equals supply) implies a price of 0! So long as the price of a rainbow is positive, the firm would rather shut down.

Now suppose that the firm can put a single pony out to pasture; that is, the firm has free disposal of ponies. A strategic firm can see that the market-clearing price of ponies, p_h is such that

$$p_h \approx \begin{cases} > k & \text{if } h < 1, \\ \in [0, k] & \text{if } h = 1, \\ = 0 & \text{otherwise.} \end{cases} .$$

Thus if the firm sells a single pony, it may charge up to $p_h = k$ and clear the market. Given that the price of a rainbow is less than k, this will obtain the firm positive profits, strictly better than the profits available if free disposal is not an option. In fact, notice that without free disposal *this strategic firm's problem has no solution!* It stands to reason that in the case of a monopoly (or otherwise strategic firm) free disposal may not be innocuous at all, and may be necessary to find any solution at all to the firm's problem.

Example: neither ponies nor rainbows[2]

For a somewhat less ridiculous example, we will consider a more staid firm which produces y_2 from y_1. Its production set is given by

$$Y = \left\{(y_1, y_2) : y_1 \leq -1, y_2 \leq \sqrt{k - y_1} - \sqrt{k+1}\right\}, \quad k \geq -1.$$

Find profit-maximizing production, given prices p.

Solution: we see that the maximization problem $\max_{y \in Y} p \cdot y$ may be stated as

$$\max_y \; p \cdot y, \text{ s.t. } y_1 \leq -1 \text{ and } y_2 \leq \sqrt{k - y_1} - \sqrt{k+1}.$$

This has a Lagrangian of

$$L(y) = p_1 y_1 + p_2 y_1 + \lambda\left(\sqrt{k - y_1} - \sqrt{k+1} - y_2\right) - \mu(1 + y_1).$$

First-order conditions give us

$$\begin{aligned} \frac{\partial}{\partial y_1} &= p_1 - \frac{\lambda}{2}\sqrt{k - y_1}^{-1} - \mu, \\ \frac{\partial}{\partial y_2} &= p_2 - \lambda. \end{aligned}$$

By a rough analogue of Walras' law, the production constraint must always bind; that is, since prices are strictly positive greater production of y_2 is always better for the firm, hence the ultimate production of y_2 must meet capacity. From the first-order conditions, we have that $p_2 = \lambda$; hence we see

$$\begin{aligned} & p_1 - \frac{p_2}{2}\sqrt{k - y_1}^{-1} - \mu = 0 \\ \iff \quad & \sqrt{k - y_1} = \frac{p_2}{2(p_1 - \mu)} \\ \iff \quad & y_1 = k - \left(\frac{p_2}{2(p_1 - \mu)}\right)^2, \\ \implies \quad & y_2 = \frac{p_2}{2(p_1 - \mu)} - \sqrt{k+1}. \end{aligned}$$

[2] Formally, we issue no specification as to what y_1 and y_2 are; while they may very well be ponies and rainbows (respectively), this is decidedly unlikely.

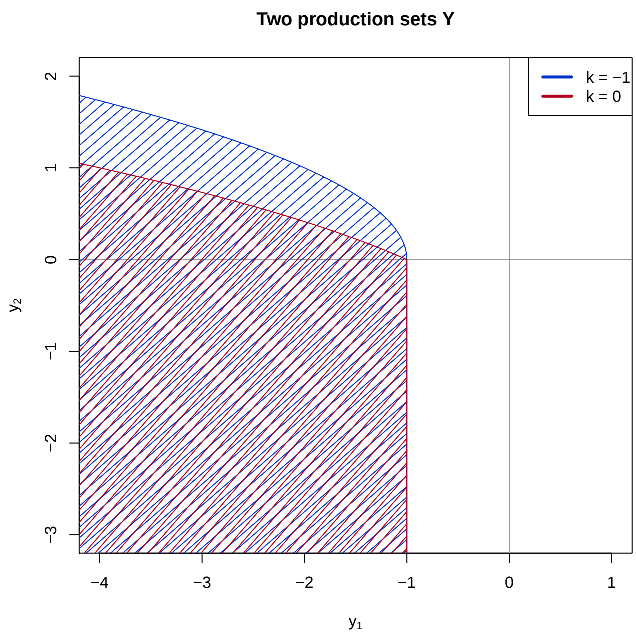

Figure 1: example production sets for $k \in \{-1, 0\}$.

Intuitively, we can see that y_1 decreases and y_2 increases as p_2 increases; since production inputs are negative, this is the correct relationship. The converse holds for the relationship with p_1.

Considering the Lagrange multiplier μ, we see that if the factor constraint on y_1 does not bind,

$$y_1 = k - \left(\frac{p_2}{2p_1}\right)^2;$$

this is consistent when

$$\begin{aligned} -1 &\geq k - \left(\frac{p_2}{2p_1}\right)^2 \\ \iff \quad k + 1 &\leq \left(\frac{p_2}{2p_1}\right)^2 \\ \iff \quad 2\sqrt{k+1} &\leq \frac{p_2}{p_1}. \end{aligned}$$

This leaves us the following expression for the (unique) profit-maximizing production plan,

$$y(p) = \begin{cases} \left(k - \left(\frac{p_2}{2p_1}\right)^2, \frac{p_2}{2p_1} - \sqrt{k+1}\right) & \text{if } 2\sqrt{k+1} \leq \frac{p_2}{p_1}, \\ (-1, 0) & \text{otherwise.} \end{cases}$$

Notice that when $k = -1$, production is *always* interior (in the sense that $y_1 < -1$) for any $p \gg 0$. This is because — in this case — there is infinite marginal production when $x = -1$; that is, some analogue of the Inada conditions holds. When $k > -1$, there is a set of price ratios which yield production at the kink in the production set. The existence of such corner solutions can be clearly seen in Figure 2.

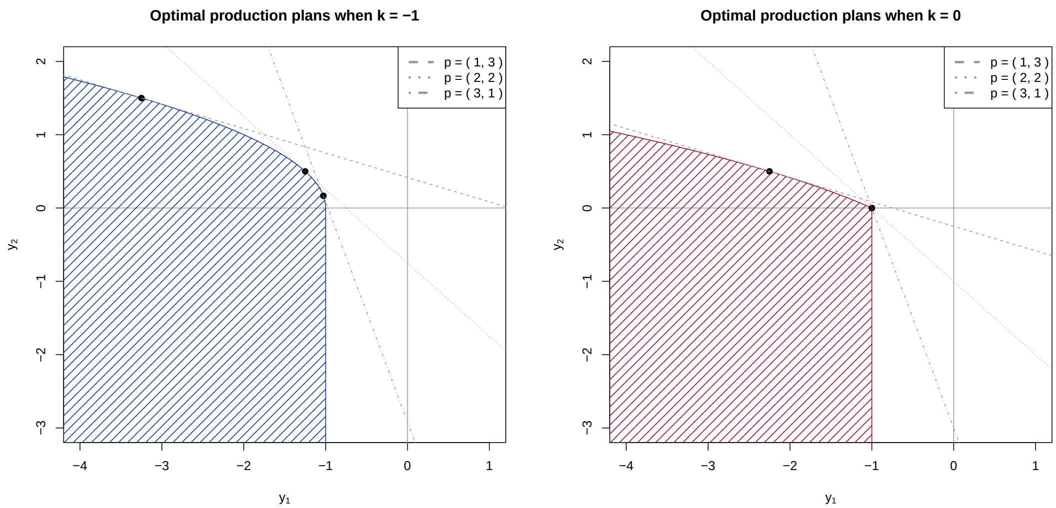

Figure 2: optimal production plans for various prices, and $k \in \{-1, 0\}$. Note the corner solutions arising when $k \neq -1$.

Example: shutdown

Suppose we have the same setup as above, except that shutdown — $y = (0, 0)$ — is now feasible. What is the firm's optimal production plan?

Solution: from above, we have an expression for the firm's optimal behavior given that production lies on the "main body" of the production set. What shutdown permits is the potential to make zero profits; you can see in Figure 2 that profits are sometimes negative in the problem without shutdown. Solving the firm's problem with shutdown is then a matter of checking whether or not profits are positive. That is, if profits from the above problem are positive, the firm is strictly better off than shutting down; if negative, then shutdown is strictly preferred, and will now be pursued.

To begin, notice that profits from the corner solution $y = (-1, 0)$ are strictly negative, $r = -p_1$. Then if we are in a case in which corner production previously would have been followed, the firm will now shut down. This simplifies our analysis, since we now only care about interior behavior.

Profits from the interior solution are given by

$$\begin{aligned} p_1\left(k - \left(\frac{p_2}{2p_1}\right)^2\right) + p_2\left(\frac{p_2}{2p_1} - \sqrt{k+1}\right) &= p_1 k - \frac{p_2^2}{4p_1} + \frac{p_2^2}{2p_1} - p_2\sqrt{k+1} \\ &= p_1 k + \frac{p_2^2}{4p_1} - p_2\sqrt{k+1}. \end{aligned}$$

Then profits are positive when

$$\begin{aligned} & p_1 k + \frac{p_2^2}{4p_1} - p_2\sqrt{k+1} > 0 \\ \iff \quad & \left(\frac{p_1}{p_2}\right) k + \frac{1}{4}\left(\frac{p_2}{p_1}\right) - \sqrt{k+1} > 0 \\ \iff \quad & \rho k + \frac{1}{4}\rho^{-1} - \sqrt{k+1} > 0 \\ \iff \quad & \rho^2 k - \rho\sqrt{k+1} + \frac{1}{4} > 0. \end{aligned}$$

Here, $\rho = \frac{p_1}{p_2}$ is the price ratio. When $k = 0$, the solution is[3]

$$\rho < \left(4\sqrt{k+1}\right)^{-1} = \frac{1}{4}.$$

That is, p_1 must be significantly small relative to p_2. In all other cases — $k \neq 0$ — the above is a quadratic equation; we find zeros of

$$\rho = \frac{\sqrt{k+1} \pm \sqrt{k+1-k}}{2k} = \frac{\pm 1 + \sqrt{k+1}}{2k}.$$

Two cases arise:

(a) $k < 0$; then the parabola is downward-facing, and we want ρ on the interior. This gives us

$$\rho \in \left(\frac{1+\sqrt{k+1}}{2k}, \frac{-1+\sqrt{k+1}}{2k}\right) \quad \text{(remember, } k < 0\text{)}.$$

Since the price ratio must always be positive, this implies

$$\rho \leq \frac{-1+\sqrt{k+1}}{2k}.$$

(b) $k > 0$; then the parabola is upward-facing, and we want ρ on the exterior. Then we want

$$\rho < \frac{-1+\sqrt{k+1}}{2k} \quad \text{or} \quad \rho > \frac{1+\sqrt{k+1}}{2k}.$$

Notice that the right-hand solution is counterintuitive: the price of y_1 must be significantly *large* relative to the price of y_2. We then check this against the condition for an interior solution,

$$2\sqrt{k+1} \leq \frac{p_2}{p_1} \quad \Longleftrightarrow \quad \rho \leq \left(2\sqrt{k+1}\right)^{-1}.$$

Putting the right-hand side of both inequalities together, we find

$$\begin{aligned} & \left(2\sqrt{k+1}\right)^{-1} > \frac{1+\sqrt{k+1}}{2k} \\ \Longleftrightarrow \quad & k > \sqrt{k+1} + k + 1 \\ \Longleftrightarrow \quad & 0 > 1 + \sqrt{k+1}. \end{aligned}$$

Since this is a contradiction, the right-hand side is a contradiction of an interior solution. Thus for positive profits we must have

$$\rho < \frac{-1+\sqrt{k+1}}{2k}.$$

We may now piece all of these together. Since the firm will not produce when profits are not positive, we obtain a production function of

$$y(p) = \begin{cases} \left(k - \left(\frac{p_2}{2p_1}\right)^2, \frac{p_2}{2p_1} - \sqrt{k+1}\right) & \text{if } k = 0 \quad \text{and} \quad \frac{p_1}{p_2} < \frac{1}{4}, \\ \left(k - \left(\frac{p_2}{2p_1}\right)^2, \frac{p_2}{2p_1} - \sqrt{k+1}\right) & \text{if } k \neq 0 \quad \text{and} \quad \frac{p_1}{p_2} < \frac{-1+\sqrt{k+1}}{2k}, \\ (0, 0) & \text{otherwise.} \end{cases}$$

[3]This can be seen in Figure 2, in that each of the price lines passes below the origin.

Assets and insurance

In lecture, we briefly discussed a stripped-down version of asset pricing[4]. In this simplified model, there are two dates, $t \in \{1, 2\}$, referred to as the initial and terminal dates, respectively[5]. At the initial date the endowment is known, but at the terminal date there is some set S of states which arise with some probability; of course, if we had more than two dates, we would likely want to see randomness at all dates but the initial date, but in the two-date case this is uniquely the terminal date. S may be arbitrarily large — even uncountable, as in a continuous distribution — but for simplicity, we generally assume that S has two states, $S = \{1, 2\}$; convention gives that $\Pr(s = 1) = \pi$, where s is the realization of the state of the world.

Notation for the consumer is that e_0 is the endowment at the initial date, and e_1 and e_2 are the endowments at the terminal date in states 1 and 2, respectively. Corresponding to these subscripts we also have consumption c_0, c_1, c_2, defined similarly. The problem in an asset pricing problem is generally that of a consumer attempting to mitigate risk to some extent, according to budget $w = p_0e_0 + p_1e_1 + p_2e_2$. Utility is generally the (discounted) sum of expected utility over the dates in the model.

There are a few special cases of this setup to which we regularly refer:

- *Exchange-only.* Multiple agents work to consumption smooth by trading amongst themselves at market prices determined in the standard Walrasian way.
- *Asset pricing.* There is a risk-free asset which pays off at rate R and a risky asset which pays of ℓ in state 1 and h in state 2; we solve the consumer's problem at a given price vector. Alternatively, we price the risky asset so that there is no arbitrage.
- *Insurance.* There are multiple assets representing consumption in particular terminal states; we solve the consumer's problem at a given price vector, and the consumer has the option to sell future consumption back to the insurer in the form of buying a negative quantity of insurance.

Depending on your view of economics, one or the other of these is more interesting. We will see plenty of other Walrasian equilibrium computations, so I'll tend to avoid those models in asset pricing (unless requested); Bill has taught asset pricing in the past, and I would not be surprised if he focuses on the second point. For my part, I find proper insurance the most intuitive discussion of the concept. Fortunately, the methods are all very similar, so we are well enough off solving one type of problem at the expense of the others.

Actuarial fairness

Actuarial fairness is an assumption on price ratios which is particularly relevant in setups with insurance firms. An asset price vector is *actuarially fair* if any asset portfolio yields zero expected profits for the insurer. In particular, if one unit of a simple asset k pays off y_k in state k, actuarial fairness in a world without discounting implies

$$p_k - \pi_k y_k = 0,$$

where π_k is the probability of state k arising. As a rule, we will generally assume that $y_k = 1$ — the asset pays off 1 in state k — so that comparisons between asset prices are intuitive and meaningful; this assumption gives rise to a very clean expression for the most useful form of actuarial fairness,

$$p_k = \pi_k.$$

[4] As Bill calls it, "baby asset pricing."

[5] Depending on your opinion of 1-indexing, we could also say $t \in \{0, 1\}$. Generally speaking, in this model there is no confusion since we have a very small number of dates and states, but in a larger model this should probably be made reader-friendly.

That is, prices are equal to probabilities. Somewhat more generally, we need that prices are *proportional* to probabilities, or that price ratios equal probability ratios. This allows us to scale prices arbitrarily, ensuring the proper homogeneity of degree zero of Walrasian demand.

Of course, more complicated assets will be more difficult to price, at least in terms of the equations specifying their prices. Further, the introduction of a discount rate for the insurer — representing, say, a prevailing market interest rate — will have downstream implications for the precise nature of actuarial fairness[6].

Why do we assume actuarial fairness in insurance markets? One simple yet indirect answer is that it makes algebra simpler. More realistically, though, according to Walrasian firm theory, we expect firms to profit-maximize given prices; if the insurance market is constant returns to scale — which we get from the fact that the insurer is risk-neutral — then insurance may be scaled up indefinitely[7]. If positive profits arise from any contract, the insurer will be sure to supply an infinite amount; if negative profits arise, the ability of the insurer to buy contracts will have the insurer purchasing an infinite amount. The only set of prices which will ensure the ability of markets to clear will have zero expected profits, or actuarially fair prices.

Below, we discuss two examples of how an insurer might make profits with prices being roughly actuarially fair. In the first case, we are not violating the argument that supply will go to infinity, since a deductible matters only on the extensive margin, and does not represent profits per-unit beyond the first contract. The second example retains the spirit of actuarially fair pricing while blatantly violating our argument about bounded supply and demand above; however, since we are not solving for general equilibrium in this case but for consumer demand for the assets, we will paper over this inconsistency.

Example: deductible

Suppose there are two dates and states. At the initial date, the agent has endowment e; at the terminal date, the agent endowment 0 with probability π (state 1) and again has endowment e with probability $(1-\pi)$ (state 2). An insurance company offers forward contracts for consumption in one state or the other at actuarially fair prices; the agent is free to buy or sell these contracts. The agent's intertemporal utility is in expectation and is not discounted, i.e.,

$$u(c_0, c_1, c_2) = \ln(c_0 + k) + \pi \ln(c_1 + k) + (1-\pi)\ln(c_2 + k), \quad k > 0.$$

To the agent's disapproval, the insurer does not pay out the full value of the contract in the event that it is to pay the agent (when the agent has purchased a forward contract for this state); prior to payment, it retains a deductible b, or the entirety of the contract if less than b. There is no symmetry to this offer, and in the event that the agent is to make a payment to the insurer the entire contract is paid. When does the insurer make positive profits?[8]

Solution: we attack this question in three stages. For notation, let x_1 and x_2 represent the net amount of insurance purchased in states 1 and 2, respectively.

(i) *If $x_1 > 0$, then $x_1 > b$.*

Notice that $c_2 = w + x_2$, and $c_0 = w - p_1 x_1 - p_2 x_2$. Suppose $x_1 \in (0, b]$; then $c_1 = x_1 - \min\{x_1, b\} = 0$. We can see that

$$\ln(w - p_1 x_1 - p_2 x_2 + k) + \pi \ln k + (1-\pi)\ln(w + x_2) < \ln(w - p_2 x_2 + k) + \pi \ln k + (1-\pi)\ln(w + x_2).$$

[6] It is not complicated to perform this exercise: ask what expected profits are in a world with a discount rate, then solve for zero profits.

[7] This may ignore troubles such as overhead from employing underwriters, but if we ignore this detail and assume that the insurer is perfectly able to pay or collect on contracts with no frictions, the scalability is clear.

[8] Notice that this differs from the setup given in section. In section, we papered over point (i) below. However, to be more rigorous we will need to formally establish why $x_1 > 0$ implies $x_1 > b$; to do so will require that utility be well-defined at all points. When $k = 0$ — as in section — the argument that some other amount of investment is better is not strictly true, albeit quite intuitive.

Thus if $x_1 \in (0, b)$, an investment of $x_1 = 0$ is strictly preferred. Notice that we have said nothing yet about the signs of x_1 and x_2, only that if x_1 is strictly positive it must strictly cover the deductible.

(ii) *The insurer makes a profit only if $x_1 > 0$ or $x_2 > 0$.*

To assist computation, define "per-state" profits[9] by

$$r_j = \begin{cases} (p_j - \pi_j)x_j & \text{if } x_i \leq 0, \\ (p_j - \pi_j)x_j + \pi_j \min\{x_j, b\} & \text{otherwise.} \end{cases}$$

Since prices are actuarially fair, we know $p_j = \pi_j$ in each state j. Hence profits may be spelled out as

$$r_j = \begin{cases} 0 & \text{if } x_i \leq 0, \\ \pi_j \min\{x_j, b\} & \text{otherwise.} \end{cases}$$

In light of point (i) above, the latter case effectively becomes $\pi_j b$, but this is not how it enters the firm's problem. Thus in our specification, we see that the firm will turn a profit of $\pi_j \min\{x_j, b\}$ in any state j in which $x_j > 0$, and hence the firm will obtain positive profits only if one of $x_1 > 0$ or $x_2 > 0$ holds.

(iii) *When is it the case that $x_1 > 0$?*

This is a matter of solving a standard first-order conditions problem. The consumer's problem is given by

$$\begin{aligned} &\max_{c,x} \ \ln(c_0 + k) + \pi \ln(c_1 + k) + (1 - \pi) \ln(c_2 + k), \\ \text{s.t.} \quad & c_0 = w - p_1 x_1 - p_2 x_2, \\ & c_1 = x_1 - 1[x_1 > 0]b, \\ & c_2 = w + x_2 - 1[x_2 > 0]b. \end{aligned}$$

The condition on c_1 is derived from part (i) above; the condition on c_2 follows analogously. Thus the Lagrangian is given by

$$L(x) = \ln(w - p_1 x_1 - p_2 x_2 + k) + \pi \ln(x_1 - 1[x_1 > 0]b + k) + (1 - \pi) \ln(w + x_2 - 1[x_2 > 0]b + k).$$

First-order conditions yield

$$\begin{aligned} \frac{\partial}{\partial x_1} &= -\frac{p_1}{w - p_1 x_1 - p_2 x_2 + k} + \frac{\pi}{x_1 - 1[x_1 > 0]b + k}, \\ \frac{\partial}{\partial x_2} &= -\frac{p_2}{w - p_1 x_1 - p_2 x_2 + k} + \frac{1 - \pi}{w + x_2 - 1[x_2 > 0]b + k}. \end{aligned}$$

Actuarial fairness implies the following equalities,

$$\begin{aligned} w - p_1 x_1 - p_2 x_2 + k &= x_1 - 1[x_1 > 0]b + k \\ w - p_1 x_1 - p_2 x_2 + k &= w + x_2 - 1[x_2 > 0]b + k \\ \rightsquigarrow \qquad x_1 - 1[x_1 > 0]b &= w + x_2 - 1[x_2 > 0]b. \end{aligned}$$

We can see that this implies a few cases (importantly, though, we cannot have $x_1 < 0$ while $x_2 > 0$); we will check each of these individually.

[9] Generally speaking, profits are denoted by π. In the setup we've been using, π is a probability measure. Another standard probability notation is p, which would of course be confused with prices; thus we will have to deal with this little discrepancy by using a nonstandard notation for profits, r.

(a) $x_1 > 0, x_2 < 0$. This is the case in which insurance is bought against state 1, while insurance is sold from state 2. Given intuition about consumption smoothing, this appears to be the most likely case. With $x_1 > 0$, $x_2 < 0$ we find that $x_1 = w + x_2 + b$, or $x_2 = x_1 - b - w$. Hence we may substitue in to find

$$\begin{aligned} & w - p_1 x_1 - p_2 x_2 + k = x_1 - b + k \\ \implies \quad & w - \pi x_1 - (1-\pi)(x_1 - b - w) = x_1 - b \\ \implies \quad & w + b + (1-\pi)b + (1-\pi)w = 2x_1 \\ \implies \quad & x_1 = \frac{(2-\pi)(w+b)}{2}. \end{aligned}$$

Since $x_1 > 0$ implies $x_1 > b$, we now check

$$\begin{aligned} & x_1 > b \\ \iff \quad & (2-\pi)(w+b) > 2b \\ \iff \quad & w + (1-\pi)w > \pi b \\ \iff \quad & w + p_2 w > p_1 b. \end{aligned}$$

That is, coverage is purchased when the total available wealth across dates (discounted according to expectation/prices) is at least sufficient to purchase coverage which extends past the (expected) deductible b. We should also check that x_2 is negative,

$$x_2 = x_1 - b - w = \frac{-\pi(w+b)}{2} < 0,$$

so our solution is consistent.

(b) $x_1 > 0, x_2 > 0$. We then have that $x_1 = x_2 + w$, or $x_2 = x_1 - w$. Hence we may substitute in to find

$$\begin{aligned} & w - p_1 x_1 - p_2 x_2 + k = x_1 - b + k \\ \implies \quad & w - \pi x_1 - (1-\pi)(x_1 - w) = x_1 - b \\ \implies \quad & w + b + (1-\pi)w = 2x_1 \\ \implies \quad & x_1 = \frac{(2-\pi)w + b}{2}, \\ \implies \quad & x_2 = \frac{-\pi w + b}{2}. \end{aligned}$$

Since $x_2 > 0$ implies $x_2 > b$, we now check

$$\begin{aligned} & x_2 > b \\ \iff \quad & \frac{-\pi w + b}{2} > b \\ \iff \quad & -\pi w > b, \end{aligned}$$

which is a clear contradiction; hence we cannot have $x_1 > 0, x_2 > 0$.

(c) $x_1 < 0, x_2 < 0$. We then have again that $x_1 = x_2 + w$, or $x_2 = x_1 - w$. Hence we may substitute in to find

$$\begin{aligned} & w - p_1 x_1 - p_2 x_2 + k = x_1 + k \\ \implies \quad & w - \pi x_1 - (1-\pi)(x_1 - w) = x_1 \\ \implies \quad & x_1 = \frac{(2-\pi)w}{2} > 0. \end{aligned}$$

We do not need to solve for x_2, since this is already a contradiction. Hence we cannot have $x_1 < 0, x_2 < 0$.

We thus have a unique possible set of insurance trades, $x_1 > 0$ and $x_2 < 0$. Our last step is to check that trade is strictly preferred to no trade,

$$\begin{aligned}
&\ln\left(w - \pi\left(\frac{(2-\pi)(w+b)}{2}\right) + (1-\pi)\left(\frac{\pi(w+b)}{2}\right) + k\right) + \pi\ln\left(\frac{(2-\pi)(w+b)}{2} - b + k\right) \\
&\quad + (1-\pi)\ln\left(w - \pi\left(\frac{w+b}{2}\right) + k\right) \\
&= \ln\big(2w + 2k + (\pi^2 - 2\pi)(w+b) + (\pi - \pi^2)(w+b)\big) + \pi\ln\left((2-\pi)(w+b) - 2b + 2k\right) \\
&\quad + (1-\pi)\ln\left(2w + 2k - \pi(w+b)\right) - 2\ln 2 \\
&= \ln((2-\pi)w - \pi b + 2k) + \pi\ln((2-\pi)w - \pi b + 2k) + (1-\pi)\ln((2-\pi)w - \pi b + 2k) - 2\ln 2 \\
&= 2\ln\left(\frac{2w - \pi(w+b)}{2} + k\right) \\
&> \ln(w+k) + \pi\ln k + (1-\pi)\ln(w+k) \\
&= \ln\left((w+k)^{2-\pi}k^{\pi}\right)
\end{aligned}$$

This yields a rather obtuse condition,

$$b < \frac{(2-\pi)w + 2k - 2\sqrt{(w+k)^{2-\pi}k^{\pi}}}{\pi}.$$

When this condition on the deductible holds — as well as that derived in (a) — we will have positive purchase of insurance against state 1, and the insurer's profits will then be positive. Notice that, as $k \to 0$, the right-hand side goes to $\frac{(2-\pi)w}{\pi}$, which is exactly the condition obtained in (a) and in section (where we assumed $k = 0$, ignoring algebraic issues).

Example: "constant cut"

Another way an insurer might make positive profits is by scaling state prices by some constant factor c; that is, while forward-state price *ratios* remain actuarially fair, the intertemporal margins are distorted by c. Using the same setup from above, except that the deductible policy is replaced with this constant multiple policy, when does the insurer make positive profits?

Solution: we are in a slightly nicer world here, since the implicit continuity of the problem should keep us from case-analysis. The insurer's expected profits are given by

$$\begin{aligned}
(\beta p_1 - \pi)x_1 + (\beta p_2 - (1-\pi))x_2 &= (\beta - 1)p_1 x_1 + (\beta - 1)p_2 x_2 + (p_1 - \pi)x_1 + (p_2 - (1-\pi))x_2 \\
&= (\beta - 1)(\pi x_1 + (1-\pi)x_2) \quad \text{(by actuarial fairness of unscaled prices).}
\end{aligned}$$

Thus whether or not the firm makes a profit will depend on the signs and magnitudes of x_1, x_2, as well as on the magnitude of c. To put some more certainty around this, we will need to solve the consumer's problem.

The consumer's problem is given by

$$\begin{aligned}
&\max_{c,x} \; \ln(c_0 + k) + \pi\ln(c_1 + k) + (1-\pi)\ln(c_2 + k), \\
\text{s.t.} \quad & c_0 = w - \beta p_1 x_1 - \beta p_2 x_2, \\
& c_1 = x_1, \\
& c_2 = w + x_2.
\end{aligned}$$

The Lagrangian is therefore

$$L(x) = \ln(w - \beta p_1 x_1 - \beta p_2 x_2 + k) + \pi\ln(x_1 + k) + (1-\pi)\ln(w + x_2 + k).$$

First-order conditions yield

$$\frac{\partial}{\partial x_1}[L(x)] = -\frac{\beta p_1}{w - \beta p_1 x_1 - \beta p_2 x_2 + k} + \frac{\pi}{x_1 + k},$$
$$\frac{\partial}{\partial x_2}[L(x)] = -\frac{\beta p_2}{w - \beta p_1 x_1 - \beta p_2 x_2 + k} + \frac{1-\pi}{w + x_2 + k}.$$

Appealing to actuarial fairness, we know that $p_1 = \pi$ and $p_2 = (1-\pi)$, hence we can find

$$x_1 + k = w + x_2 + k \quad \Longrightarrow \quad x_1 = x_2 + w.$$

Substituting in as we usually do,

$$\begin{aligned}
& w - \beta p_1 x_1 - \beta p_2 x_2 + k = \beta x_1 + \beta k \\
\Longrightarrow \quad & w + (1-\beta)k - \beta(1-\pi)(x_1 - w) = \beta(1+\pi)x_1 \\
\Longrightarrow \quad & (1 + \beta(1-\pi))w + (1-\beta)k = 2\beta x_1 \\
\Longrightarrow \quad & x_1 = \frac{(1+\beta(1-\pi))w + (1-\beta)k}{2\beta}, \\
\Longrightarrow \quad & x_2 = \frac{(1-\beta(1+\pi))w + (1-\beta)k}{2\beta}.
\end{aligned}$$

With these values in hand, we find

$$\begin{aligned}
\pi x_1 + (1-\pi)x_2 &= \frac{(1-\beta\pi)w + (1-\beta)k}{2\beta} + \frac{1}{2\beta}\left(\pi\beta - (1-\pi)\beta\right)w \\
&= \frac{(1+\beta(\pi-1))w + (1-\beta)k}{2\beta}.
\end{aligned}$$

To sign insurer's profits, we check

$$\begin{aligned}
& \pi x_1 + (1-\pi)x_2 > 0 \\
\Longleftrightarrow \quad & (1+\beta(\pi-1))w + (1-\beta)k > 0 \\
\Longleftrightarrow \quad & \beta((\pi-1)w - k) > -(w+k) \\
\Longleftrightarrow \quad & \beta < \frac{w+k}{(1-\pi)w+k}.
\end{aligned}$$

Then we see that the insurer makes strictly positive profits so long as

$$\beta \in \left(1, \frac{w+k}{(1-\pi)w+k}\right).$$

Example: aggregate risk

Recall Bill's proof from lecture that identical aggregate endowments imply actuarially fair price ratios. Per request, we give a concrete example here of differing aggregate endowments leading to actuarially unfair prices.

Suppose we have two consumers, $i \in \{1,2\}$, each with endowment given by (e_0^i, e_1^i, e_2^i), where subscripts correspond to a two-date/state world in the usual way. The probability of state 1 at date 2 is π. Bernoulli utility for each agent is given by $u(c) = \ln c$; overall utility is in expectation, with no discounting. What is the aggregate price ratio?

Solution: to begin, we solve an individual consumer's problem, given by

$$\max_{c^i} \ \ln c_0^i + \pi \ln c_1^i + (1-\pi)\ln c_2^i, \ \text{ s.t. } p_0 c_0^i + p_1 c_1^i + p_2 c_2^i \leq p_0 e_0^i + p_1 e_1^i + p_2 e_2^i = w^i.$$

Jumping straight to first-order conditions, we obtain

$$\begin{aligned}
\frac{\partial}{\partial c_0^i}[L(c)] &= \frac{1}{c_0^i} - \lambda p_0, \\
\frac{\partial}{\partial c_1^i}[L(c)] &= \frac{\pi}{c_1^i} - \lambda p_1, \\
\frac{\partial}{\partial c_2^i}[L(c)] &= \frac{1-\pi}{c_2^i} - \lambda p_2.
\end{aligned}$$

When we substitute these implicit equations for consumption back into the budget constraint, we obtain

$$\lambda = \frac{2}{w^i}.$$

Consumption then follows the standard Cobb-Douglas pattern,

$$c_0^i = \frac{w^i}{2p_0}, \quad c_1^i = \frac{\pi w^i}{2p_1}, \quad c_2^i = \frac{(1-\pi)w^i}{2p_2}.$$

To find market prices, we simply substitute back into the market clearing constraint,

$$c_j^1 + c_j^2 = \frac{\mathbb{P}_j w^1}{2p_j} + \frac{\mathbb{P}_j w^2}{2p_j} = e_j^1 + e_j^2 \equiv E_j.$$

We can then see that

$$p_j = \frac{\mathbb{P}_j(w^1 + w^2)}{2E_j};$$

notice that this form is not quite explicit, since wealth w^i is itself a function of prices. However, this expression is sufficient to obtain the equilibrium price ratios,

$$\frac{p_j}{p_k} = \frac{\left(\frac{\mathbb{P}_j(w^1+w^2)}{2E_j}\right)}{\left(\frac{\mathbb{P}_k(w^1+w^2)}{2E_k}\right)} = \left(\frac{\mathbb{P}_j}{\mathbb{P}_k}\right)\left(\frac{E_j}{E_k}\right)^{-1}.$$

That is, equilibrium price ratios are related to the underlying probability ratios — which define actuarial fairness — but are further scaled by the inverse ratio between aggregate endowments in the two dates/states. It is immediate from this expression that when aggregate endowments are identical in all dates and states, price ratios are pinned down completely by probability ratios.