This document contains a few small proofs, hopefully explained reasonably well. Econ 11 does not require you to be able to prove things. However, for exposition's sake I find it helpful to do something a little more than just state results. If you aren't into proofs, don't worry; if they help you, then good.

Budget sets

To date, we have been concerned with agent preferences and selection from sets of available alternatives. While this framework is quite general, it lacks a certain sort of economic applicability; although people certainly make choices from sets of available things, how exactly to put this in the context of the larger macroeconomy is less than obvious.

To bridge this gap, we will need to introduce the concepts of *prices* and *wealth*. If there are ℓ commodities — so a bundle is a vector $x \in \mathbb{R}^\ell$ — then good i is associated with *price* p_i: one unit of good i costs p_i. An agent has *wealth* denoted by w, which represents his purchasing power.[1]

Given prices p, the *cost* of a particular bundle x is given by

$$c = p_1x_1 + p_2x_2 + \ldots + p_\ell x_\ell.$$

This is intuitive: p_ix_i represents the amount of wealth allocated to consuming amount x_i of good i when its price is p_i; the overall cost of the bundle is the sum of the amount of wealth allocated to each particular commodity. Equally intuitively, given prices p and wealth w, a bundle x is *affordable* if its cost is less than the available wealth; that is, when

$$p_1x_1 + p_2x_2 + \ldots + p_\ell x_\ell \leq w.$$

Using this concept, we can define the agent's *budget set* as the set of all bundles which are affordable:

$$B = \{x : p_1x_1 + p_2x_2 + \ldots + p_\ell x_\ell \leq w\}.$$

The *budget frontier* is the outer edge of the set, where everything is exactly affordable:

$$\partial B = \{x : p_1x_1 + p_2x_2 + \ldots + p_\ell x_\ell = w\}.$$

Two-dimensional examples

To keep graphs simple, it is helpful to work in a world with two commodities. As Bill has repeatedly pointed out, our theory will extend to far greater numbers of goods, but a good deal of intuition can be found in the simple, two-commodity case.

Suppose that there are two goods, x and y. The agent has wealth $w = 4$, the price of good x is $p_x = 1$, and the price of good y is $p_y = 1$. In the defining equation of the budget set, we see that we want

$$\begin{aligned} & p_xx + p_yy \leq w \\ \iff \quad & x + y \leq 4 \\ \iff \quad & y \leq 4 - x. \end{aligned}$$

Plotting the budget set is then a matter of representing this linear inequality; this is seen in Figure 1.

[1] At this level, it is okay to think of wealth as dollars. However, more generally, this is not the case: money is not useful only in its ability to buy other things, but carries some utility of its own (social cache, good feelings, etc.). The concept of wealth or purchasing power carries none of these connotations; it is purely a vehicle for acquiring utility through the consumption of other commodities.

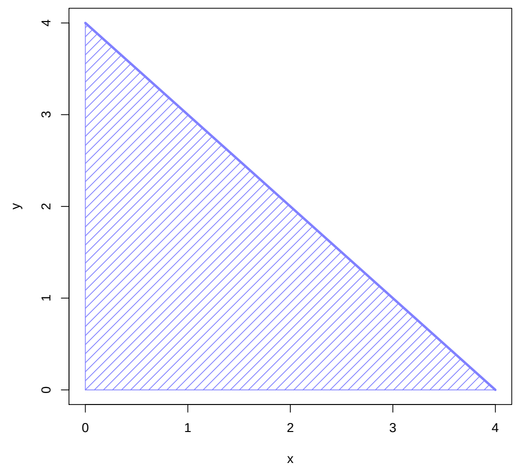

Figure 1: the agent's budget set when $(p_x, p_y, w) = (1, 1, 4)$.

More generally, we can see that leaving prices and wealth arbitrary the expression of this linear inequality would be

$$\begin{aligned} & p_x x + p_y y \leq w \\ \iff \quad & p_y y \leq w - p_x x \\ \iff \quad & y \leq \frac{w}{p_y} - \left(\frac{p_x}{p_y}\right) x. \end{aligned}$$

In particular, the slope of the budget frontier is $-\frac{p_x}{p_y}$. This will be useful later.

Special consideration should be put toward understanding where the budget frontier meets the axes (we refer to these points as *corners*). In particular, when the budget frontier meets the x-axis, it must be that y consumption is 0; all wealth is then directed toward consumption of x. With wealth w and the price of x being p_x, the amount of x which may be consumed is $x = \frac{w}{p_x}$. Similarly, along the y-axis it must be that x consumption is 0; hence $y = \frac{w}{p_y}$.

We can obtain these points from the inequality above, however they provide a useful outside check to earlier math. In particular, since we know that the budget frontier is a line, we can compute these two points and then connect them without ever directly considering the budget/affordability inequality. As I see it, this construction is slightly more intuitive than the direct algebraic solution: it is simple to consider the case of how much can be spent on just one commodity; it is also fairly clear that if you can afford A and B, then you can afford anything between A and B.

Let's return now to our earlier example with $w = 4$ and $p_x = p_y = 1$.

- What happens if $w = 2$ instead? Intuitively, we can afford less. This implies that the budget frontier shifts inward; since the price vector is unchanged, the slope will be unaffected. We can compute the intercepts as $(0, \frac{w}{p_y}) = (0, 2)$ and $(\frac{w}{p_x}, 0) = (2, 0)$. The budget set is seen in Figure 2.

- What happens if $p_x = 2$ instead? Intuitively, we can afford less x but the same amount of y. This amounts to a rotation of the budget frontier about the "anchor" on the y-axis. The x-intercept will change to $(\frac{w}{p_x}, 0) = (2, 0)$, and the budget set is pictured in Figure 2. Notice that when we increase the price of x, its intercept actually *falls*. Although this makes sense in every reasonable way — the fact that x costs more means that we can afford less of it — it may be at odds with your on-the-spot feeling for how the set shifts.

- What happens if $p_y = 2$ instead? This is identical to the above argument, but now x is unaffected while we can afford less y. The budget line rotates about the anchor on the x-axis, and the y-intercept becomes $(0, \frac{w}{p_y}) = (0, 2)$. The budget set is seen in Figure 3.

- Last, what happens if $p_x = p_y = 2$? Now, we can afford less of both x and y. To find the new intercepts, we again solve $(0, \frac{w}{p_y}) = (0, 2)$ and $(\frac{w}{p_x}, 0) = (2, 0)$. Comparing this to the first item, we can see that doubling prices is identical to halving wealth: in terms of purchasing power, both alterations reduce the overall affordable cost by a factor of $\frac{1}{2}$. In particular,

$$p_x x + p_y y \leq \frac{1}{2} w \quad \Longleftrightarrow \quad (2p_x)x + (2p_y)y \leq w.$$

This budget set is pictured in Figure 3.

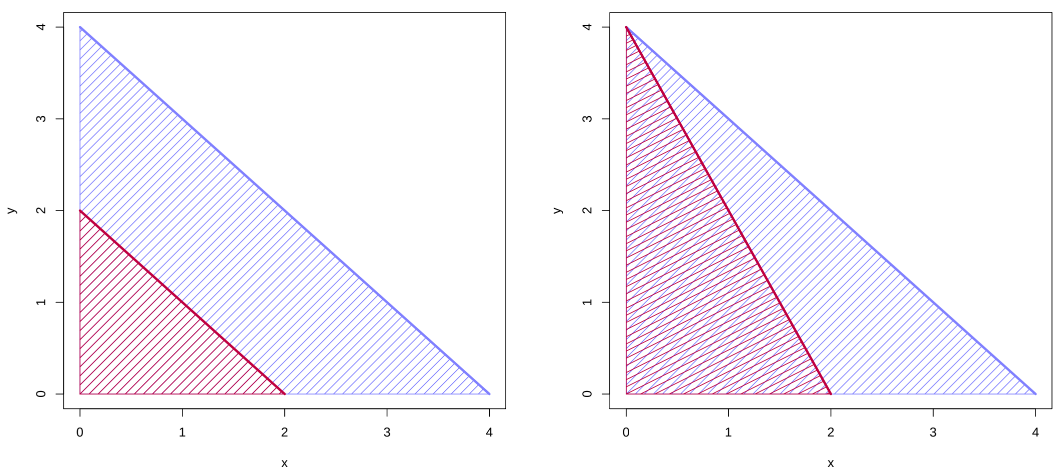

Figure 2: budget sets for $(p_x, p_y, w) = (1, 1, 2)$ (left) and $(p_x, p_y, w) = (2, 1, 4)$ (left).

Constrained optimiation: the consumer's problem

Now that we have defined the set of options available to the consumer by the budget set, we are ready to consider what choice the consumer might actually make. Since we are looking at a world where the agent can compare (via preferences) any bundle to any other, we assume that the agent will make a choice which leaves her best-off; in other words, she chooses a bundle that is preferred to any other bundle.

This is where the concept of utility truly becomes useful. If the concept of finding the most-preferred bundle seems strange, that's because it is. However, recalling that utility associates with each bundle a number which represents how preferred the bundle is to other bundles, we have a very nice statement of the consumer's problem:

$$x^* = \underset{x \in B}{\operatorname{argmax}}\, u(x).$$

To make the budget set more apparent, we usually write this as

$$x^* = \underset{x}{\operatorname{argmax}}\, u(x) \text{ s.t. } x \in B \quad \rightsquigarrow \quad x^* = \underset{x}{\operatorname{argmax}}\, u(x) \text{ s.t. } p_1 x_1 + p_2 x_2 + \ldots + p_\ell x_\ell \leq w.$$

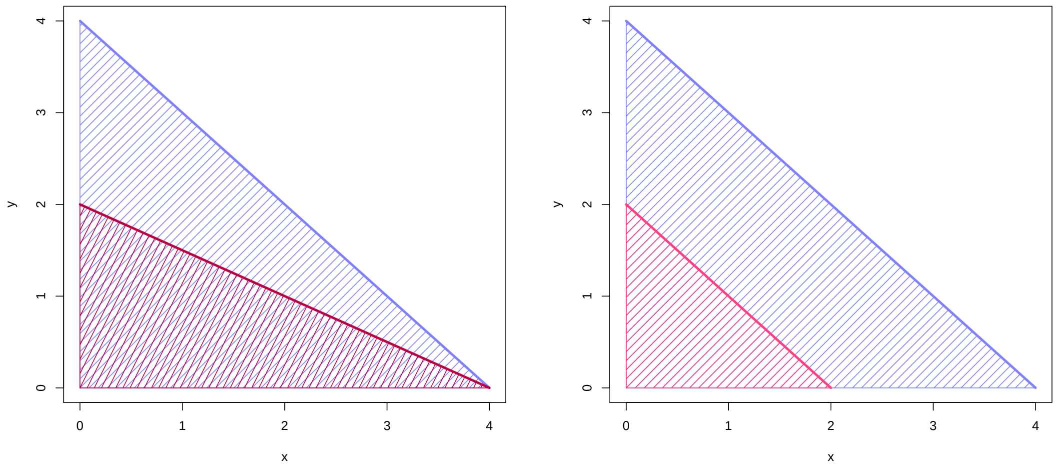

Figure 3: budget sets for $(p_x, p_y, w) = (1, 2, 4)$ (left) and $(p_x, p_y, w) = (2, 2, 4)$ (right).

Here are two useful notational pointers:

- *s.t.* This is a standard abbreviation for “such that” or “subject to.” In the above context, either works. In particular, we are looking for the maximum of $u(x)$ *subject to* the constraint that x must be within the budget set.

- *argmax*. You are accustomed to maximizing a function; we usually represent this as $\max_x u(x)$. However, the operator max returns the *value* of the function at its maximum, not the input that leads to the maximum. argmax indicates that we are not particularly interested in the maximum value, only the input which gives us the maximum value; we don’t care about utility, but we do care about the optimal bundle.

The way of reading the consumer’s problem is that the consumer will select x^* as the most-preferred bundle — that which maximizes utility — which lies within the budget set, or is affordable.

Walras’ law

Unless otherwise specified, we will assume that preferences are *increasing*, in the sense that if one bundle contains more of every good than another bundle, it is strictly preferred. More formally, if $\succeq$ represents increasing preferences and $x'_i > x_i$ for all $i \in \{1, \ldots, \ell\}$, then $x' \succ x$. In many contexts, the notion that more is better is fairly intuitive if not entirely realistic; still, this assumption drastically simplifies our analyses.

Under this assumption, we have a useful result known as *Walras’ law*: the consumer’s optimal choice x^* is such that $p_1 x_1^* + p_2 x_2^* + \ldots + p_\ell x_\ell^* = w$. Intuitively, this isn’t a stretch: if the consumer has left some purchasing power on the table (so to speak) — the bundle costs less than available wealth — then he can obtain a bundle that contains more of each commodity. Since preferences are increasing, this new bundle will leave him better off.

This is useful because it drastically simplifies the optimization problem: instead of considering where an optimum might lie within the vast expanse of the budget set, we need only consider where it might lie on the budget frontier, where $p_1 x_1 + p_2 x_2 + \ldots + p_\ell x_\ell = w$.

Solution methods

Let's consider how we might solve the agent's problem. For the duration, we will assume that there are two commodities, x and y, and that utility is given by $u(x, y) = -(x^{-1} + y^{-1})$. The agent's wealth is $w = 4$, and the prices of the commodities are $p_x = 1$, $p_y = 2$. The optimization problem above then becomes

$$\max_{x,y} u(x, y) \text{ s.t. } p_x x + p_y y \leq w \quad \rightsquigarrow \quad \max_{x,y} -(x^{-1} + y^{-1}) \text{ s.t. } x + 2y \leq 4.$$

We will solve this problem blindly in this section. There is an important caveat that we will discuss in the later section on quasilinear utility.

Direct substitution

The first method we will use is the simplest in terms of the mathematical intuition; however, it is generally the most difficult in terms of the algebra that is inevitably involved. Appealing to Walras' law, we know that the budget constraint must *bind*, $x + 2y = 4$; in particular, we know $x = 4 - 2y$. Anywhere we see an x in the objective function, then, we can replace it with this function of y. The optimization problem then becomes

$$\max_{y} -((4 - 2y)^{-1} - y^{-1}).$$

Notice that the constraint falls out! If we had kept in it place and substituted out the x term, we would have obtained

$$(4 - 2y) + 2y = 4 \quad \Longleftrightarrow \quad 4 = 4.$$

This is a general property of the method of substitution.

What's nice about this is that we now have a univariate optimization problem; from calculus we are well-aware of how to solve this: take the first derivative with respect to the variable (y) and set it equal to 0. Here, we have

$$\frac{\partial}{\partial y}[-((4 - 2y)^{-1} - y^{-1})] = -2(4 - 2y)^{-2} + y^{-2} = 0.$$

Rearranging, this is

$$\begin{aligned} & & y^{-2} &= 2(4 - 2y)^{-2} \\ &\Longleftrightarrow & y^2 &= \frac{1}{2}(4 - 2y)^2 \\ &\Longleftrightarrow & \sqrt{2}y &= 4 - 2y \\ &\Longleftrightarrow & y &= \frac{4}{2 + \sqrt{2}} \end{aligned}$$

Since we know y, and we know that $x = 4 - 2y$, we also know

$$x = 4 - \frac{8}{2 + \sqrt{2}} = \frac{4\sqrt{2}}{2 + \sqrt{2}}$$

Then the consumer's optimal consumption is given by

$$(x^*, y^*) = \left(\frac{4\sqrt{2}}{2 + \sqrt{2}}, \frac{4}{2 + \sqrt{2}} \right).$$

As a gut check, notice that x and y factor equally (in some sense) into utility. Since the price of y is higher, we should expect less of y to be consumed than x. This is indeed the case.

Marginal utility

The second approach has far more economic intuition; in the end, although it is somewhat more complex to understand it is also slightly simpler than direct substitution. Consider a point x on the budget frontier, and draw an indifference curve through it. Suppose some portion of the indifference passes strictly through part of the budget set. Then there is some y strictly within the budget set such that $y \sim x$; that is, there is some bundle on the indifference curve through x such that $p_1 y_1 + p_2 y_2 + \ldots + p_\ell y_\ell < w$.

Walras' law says that, at the optimum, the agent must be spending all of his wealth. But since y is strictly affordable, this means that we can do better! That is, there is some z in the budget set such that $z \succ y$; with $y \sim x$, this implies that $z \succ x$. Hence there is a z in the budget set which is better than x, so x cannot be an optimum.

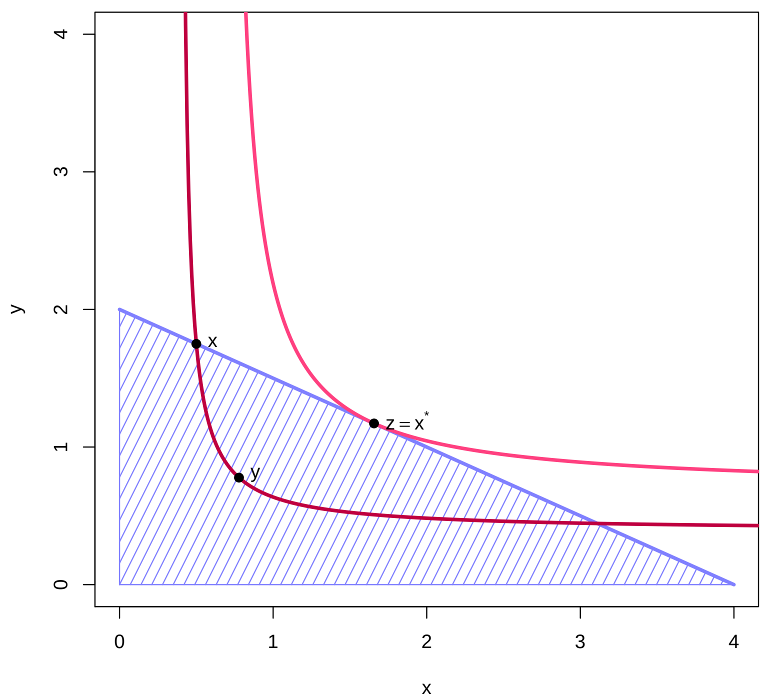

Figure 4: a graphical representation of the argument that indifference curves must meet the budget frontier only once (x, y, and z are as defined in the text).

The implication here is that it is impossible for the indifference curve through the optimum point to pass *strictly* into the budget set. Roughly speaking, this means that the indifference curve must meet the budget set at *exactly one* point.[2] Now, the budget frontier is a straight line, so it is both continuous and differentiable. As long as indifference curves are continuous (which we assume) and differentiable (most of the time, this is the case) then the concept of the indifference curve meeting the budget set at a single point implies that the indifference curve is tangent to the budget frontier, at the optimal choice x^*. The entirety of this argument is pictured in Figure 4.

How can we use this fact? In two dimensions, it is easy to find the slope of the budget frontier: we know that the frontier passes through $(0, \frac{w}{p_y})$ and $(\frac{w}{p_x}, 0)$; thus this line has rise $-\frac{w}{p_y}$ and run $\frac{w}{p_x}$, hence slope $-\frac{p_x}{p_y}$.

[2] There is an important special case in which this does not hold. Read the section on linear utility below for some idea.

Determining the slope of the indifference curve is somewhat more involved, but simple once you see it. We are looking to determine $\frac{dy}{dx}$ along the indifference curve. We can approach this the following way:

$$\frac{\frac{du}{dx}}{\frac{du}{dy}} = \left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)^{-1} = \left(\frac{du}{dx}\right)\left(\frac{dy}{du}\right) = \frac{dy}{dx}.$$

For technical reasons, it is necessary to flip the sign; so the slope of the indifference curve is $-\frac{du/dx}{du/dy}$.

We define the derivative $\frac{du}{dx}$ to be the *marginal utility of* x, and denote it MU_x. Here, marginal utility denotes the scale by which utility increases with a small increase in x. In most places in economics, we use the word *margin* instead of *derivative*; the reason is historical and is not worth going into at the moment.

We then phrase the location of the optimal bundle as

$$-\frac{\text{MU}_x}{\text{MU}_y} = -\frac{p_x}{p_y} \iff \frac{\text{MU}_x}{p_x} = \frac{\text{MU}_y}{p_y}.$$

If there are more than two commodities, this equivalence will hold across all goods; that is,

$$\frac{\text{MU}_{x_1}}{p_1} = \frac{\text{MU}_{x_2}}{p_2} = \ldots = \frac{\text{MU}_{x_\ell}}{p_\ell}.$$

Consider it this way: the marginal utility per unit cost must be identical across all items. If it wasn't, we could sell a little bit of an item with low marginal utility per unit cost (gaining some wealth without sacrificing much utility) and invest it in an item with high marginal utility per unit cost (spending the same amount of wealth, but gaining more utility).

What is nice about this solution approach is that there is no substitution up-front; this makes derivatives much simpler. Returning to the CES case above, we know

$$\text{MU}_x = \frac{\partial u}{\partial x} = x^{-2},$$
$$\text{MU}_y = \frac{\partial u}{\partial y} = y^{-2}.$$

Setting marginal utilities per unit cost equal, we have

$$\frac{x^{-2}}{p_x} = \frac{y^{-2}}{p_y} \implies p_x x^2 = p_y y^2 \implies y = x\sqrt{\frac{p_x}{p_y}} = x\sqrt{\frac{1}{2}}$$

We can find optimal consumption by plugging back into the budget constraint,

$$\begin{aligned} && p_x x + p_y y &= w \\ \rightsquigarrow && x + 2y &= 4 \\ \implies && x + x\sqrt{2} &= 4 \\ \implies && x &= \frac{4}{\sqrt{2}+1} = \frac{4\sqrt{2}}{2+\sqrt{2}} \end{aligned}$$

Again applying the budget constraint, we know $y = \frac{1}{2}(4 - x)$, hence

$$y = \frac{1}{2}\left(4 - \frac{4\sqrt{2}}{2+\sqrt{2}}\right) = \frac{4}{2+\sqrt{2}}.$$

Optimal consumption is then given by

$$(x^*, y^*) = \left(\frac{4\sqrt{2}}{2+\sqrt{2}}, \frac{4}{2+\sqrt{2}}\right).$$

Notice that this exactly coincides with the solution from direct substitution, so we are doing something right.[3]

Lagrange multipliers

The last method is a more robust method of dealing with strange cases; this approach is more technical and you should feel free to skip this section. Inasmuch as I can't predict the future, this method may or may not prove useful for Econ 11. For completeness it is included here.

Consider the optimization problem we are solving,

$$\max_{x,y} u(x,y) \text{ s.t. } p_x x + p_y y \leq w.$$

We unify the constraint and the objective through a *Lagrange multiplier* λ, to obtain a new objective function

$$\mathscr{L}(x,y,\lambda) = u(x,y) + \lambda(w - (p_x x + p_y y)).$$

To solve the optimization, take the first derivative of $\mathscr{L}$ with respect to each of its three parameters, and set them ≤ 0. In the case of the CES example above with $u(x,y) = -(x^{-1} + y^{-1})$ and $p_x = 1$, $p_y = 2$, this is

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial x} &: x^{-2} - \lambda p_x \leq 0, \\ \frac{\partial \mathscr{L}}{\partial y} &: y^{-2} - \lambda p_y \leq 0, \\ \frac{\partial \mathscr{L}}{\partial \lambda} &: w - p_x x - p_y y \leq 0. \end{aligned}$$

Walras' law tells that the third inequality holds with equality, or that it *binds*. The first two inequalities we will for now assume hold with equality. So our system is

$$\begin{aligned} x^{-2} - \lambda p_x = 0 \qquad &\Longrightarrow \qquad \lambda = \frac{1}{p_x x^2}, \\ y^{-2} - \lambda p_y = 0 \qquad &\Longrightarrow \qquad \lambda = \frac{1}{p_y y^2}, \\ & \qquad\qquad w = p_x x + p_y y. \end{aligned}$$

If we equate the first and second equations, we have

$$\frac{1}{p_x x^2} = \frac{1}{p_y y^2},$$

which is identical to the method using marginal utility! With this in mind, it is not clear here what this more-involved method gains us; more on that later.

Example: quasilinear utility

Suppose that utility is quasilinear, $u(x,y) = x + \ln y$. Wealth is $w = 5$ and prices are $p_x = 1$, $p_y = 3$. What is optimal consumption?

Applying the marginal utility approach, we see

$$\text{MU}_x = \frac{\partial u}{\partial x} = 1, \quad \text{MU}_y = \frac{\partial u}{\partial y} = \frac{1}{y}.$$

[3] To be honest, I solved this three times before I corrected all my algebra mistakes. The methods are a useful check against one another; however, the end lesson is *be careful* about your math!

It follows that

$$\frac{\mathrm{MU}_x}{p_x} = 1 = \frac{1}{3y} = \frac{\mathrm{MU}_y}{p_y}.$$

Immediately, then, we see that $y = \frac{1}{3}$. Substituting into the budget constraint under Walras' law, we have

$$p_x x + p_y y = w \quad \Longrightarrow \quad x + 3y = 5 \quad \Longrightarrow \quad x = 4.$$

Optimal consumption is then

$$(x^*, y^*) = (4, 1).$$

Corner solutions

What happens if we drastically cut the agent's wealth to $w = \frac{1}{2}$? Notice that the equation given by equal marginal utility per unit cost is not affected by wealth, so we will still have $y = \frac{1}{3}$. However, when we substitute back into the budget constraint, we find

$$x + 3y = \frac{1}{2} \quad \Longrightarrow \quad x = -\frac{1}{2}.$$

Since we require that consumption of any good is weakly positive, this won't work!

We are in a case that we call a *corner solution*. When this happens (in Econ 11), the consideration is, "To equate marginal utilities per unit cost, I need to consume negative x. Since this is not possible, I should spend all my wealth on y." So when this arises, we let $x^* = 0$ and $y^* = \frac{w}{p_y}$. In this case,

$$(x^*, y^*) = \left(0, \frac{1}{6}\right).$$

To double-check, notice that the marginal utility of x when consumption of x is 0 is $\mathrm{MU}_x = 1$. The marginal utility of y when consumption of y is $\frac{1}{6}$ is $\mathrm{MU}_y = 6$. Per unit cost, we have

$$\frac{\mathrm{MU}_x}{p_x} = 1, \quad \frac{\mathrm{MU}_y}{p_y} = 2.$$

That is, we would *like* to give up some x to purchase more y, but we cannot since we cannot consume negative amounts of a good. When wealth is large enough, this is not a problem; however, when wealth is small we need to make some tough choices.

The moral here is that you should *always* check the signs of your optimal consumptions. If consumption of one good is negative, you will need to address the situation as we did here.

Example: CES utility

Since we have already solved a CES example above, we will not do so here. However, it is worth considering *why* we were able to so brazenly chug through the CES problem without so much as considering corner solutions like those found in the quasilinear example.

Lack of corner solutions: the Inada conditions

The rationale is fairly direct, and involves a portion of some conditions (relating to macroeconomics) issued by Inada. In particular, if the marginal utility of a particular good is *infinite* when consumption of the good is 0, we cannot see 0 of this good being consumed at the optimum. More formally,

$$\mathrm{MU}_{x_i} = +\infty \text{ when } x_i = 0 \quad \Longrightarrow \quad x_i^* > 0.$$

Why is this? Recall the marginal utility method of solution: at the optimum, we should have that marginal utility per unit cost is equal across all goods (or the marginal utility of some good is small everywhere in relation to its price, so we consume none of it). If the marginal utility of a good is infinite when consumption of the good is 0, there is no way that its marginal utility per unit cost can equal any other good's! That is, if the marginal utility of a good is infinite, it is certainly utility-improving to invest a small amount of wealth in its consumption.

So if this condition — the Inada condition — is satisfied for all goods, we know that no good will have 0 consumption. In the CES case above, we can check:

$$\begin{aligned} \text{MU}_x &= x^{-2} &&\implies \text{MU}_x|_{x=0} = +\infty, \\ \text{MU}_y &= y^{-2} &&\implies \text{MU}_y|_{y=0} = +\infty. \end{aligned}$$

The Inada conditions hold for both goods, so we should see positive consumption of both. This means that there are no corner solutions possible! Usefully, Cobb-Douglas utility also has this property.

On a graphical level, remember that the marginal utility argument arises from the tangency of the indifference curve to the budget frontier. If marginal utility of either good goes to $+\infty$ as consumption of that good goes to 0, the slope of the indifference curve goes to $+\infty$ (near the y-axis) or 0 (near the x-axis). For this to be the slope of the budget frontier, $\frac{p_x}{p_y}$, we effectively need either p_x or $p_y = 0$, or p_x or $p_y = +\infty$. Since we assume prices are both positive and finite, this cannot be the case. Hence we will always consume away from the axes, if not by very much.

On plotting demand

Suppose that $p_x = 1$, but p_y and w are both variables. Often, we are concerned with how optimal consumption changes with wealth and price levels. Appealing to marginal utility approach to the CES setup above, we want

$$\frac{\text{MU}_x}{p_x} = \frac{\text{MU}_y}{p_y} \implies x^{-2} = \frac{y^{-2}}{p_y} \implies y = x\sqrt{\frac{1}{p_y}}.$$

Substituting into the budget constraint, we have

$$p_x x + p_y y = w \implies x + x\sqrt{p_y} = w \implies x = \frac{w}{1+\sqrt{p_y}}.$$

In turn, this gives

$$\frac{w}{1+\sqrt{p_y}} + p_y y = w \implies p_y y = \frac{w\sqrt{p_y}}{1+\sqrt{p_y}} \implies y = \frac{w}{p_y + \sqrt{p_y}}.$$

Optimal demand is then

$$(x^*, y^*) = \left(\frac{w}{1+\sqrt{p_y}}, \frac{w}{p_y+\sqrt{p_y}}\right).$$

How do we express this on a graph? You may recall parametric functions from a long time ago; you may not: they were the functions where you plotted both x and y against some other variable usually t. Plotting optimal consumption is like that: holding p_y or w fixed, trace out the (x^*, y^*) bundles as a function of the non-constant variable. Some examples are pictured in Figure 5.

Example: linear utility

You would think that the linear case would be the easiest; you would be wrong. There is an intuitive graphical approach to this case, but as I am running out of time we will approach it purely mathematically. As we will

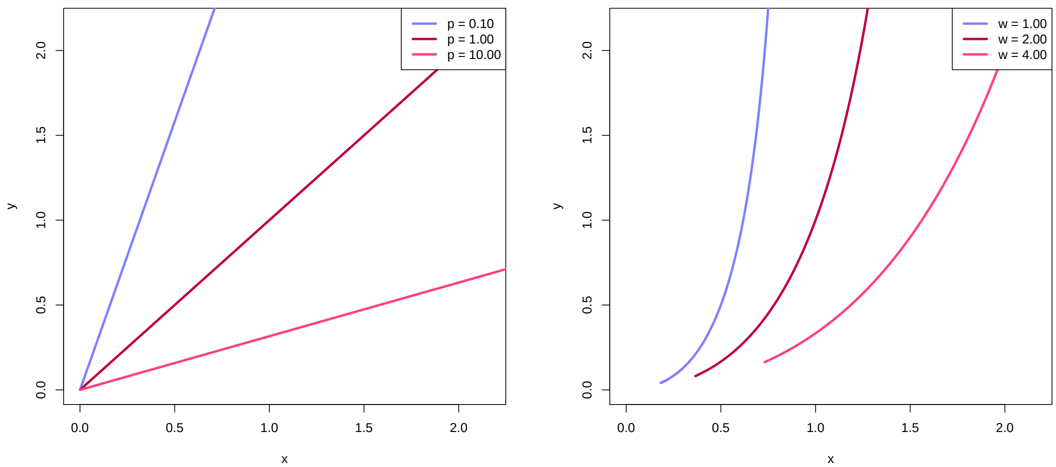

Figure 5: optimal consumption (x^*, y^*) as a function of w with p_Y fixed (left) and as a function of p_Y with w fixed (right).

use Lagrange multipliers in this case, you are not required to understand these concepts just yet; they are, however, still useful.

As a final example, we consider the case of linear utility. The difficulty with linear utility is that, since marginal utility is constant, the it is possible tha the entire budget frontier lies along the same indifference curve! To address this, we need to use the Lagrange multiplier approach. Suppose that we have utility function $u(x, y) = x + y$, with prices $p_x = 1$, $p_y = 2$, and wealth $w = 4$. The constrained optimization problem is

$$\max_{x,y} x + y \text{ s.t. } x + 2y \leq 4.$$

When we rephrase this as a Lagrangian, we have

$$\mathscr{L}(x, y, \lambda) = x + y + \lambda(4 - (x + 2y)).$$

First-order conditions give

$$\frac{\partial \mathscr{L}}{\partial x} : 1 - \lambda \leq 0,$$
$$\frac{\partial \mathscr{L}}{\partial y} : 1 - 2\lambda \leq 0,$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} : 4 - (x + 2y) \leq 0.$$

By Walras' law, we know that consumption must lie strictly on the budget frontier, so the last inequality binds: $4 - (x + 2y) = 0$. Notice that the first two inequalities become

$$\lambda \geq 1, \quad \lambda \geq \frac{1}{2}.$$

Although $\lambda = 1$ satisfies both of these inequalities, it is obvious that $\lambda = \frac{1}{2}$ will not. Hence at any solution, the second inequality must be *slack*: it must hold with strict inequality and not with equality.

The Lagrange multiplier approach tells us that a slack inequality will have consumption in that dimension equal to 0.[4] It follows then that consumption of y is 0; we then spend all of our wealth on x, so

$$(x^*, y^*) = (4, 0).$$

[4] This is not the entire formal truth of the matter, but for Econ 11 it is sufficient.

Why is this result intuitive? Notice that x and y enter utility perfectly additively. However, the costs of x and y are such that, to consume one unit of y we must forego 2 units of x! Since we care equally about it, this would be an incredibly dumb trade-off to make. It follows that we should consume zero y and only x.

Follow-up: *what changes if $p_x = 2$ and $p_y = 1$? What if $p_x = p_y = 1$? For the latter, draw the budget set and the indifference curves if you get confused; the implications of the math are not obvious!*