This is a draft; email me with comments, typos, clarifications, etc.

Efficient mechanisms

This comment regards Proposition 11.2-4 in *Essential Microeconomics*. In problem set 2, question 3, we were asked whether a better mechanism could exist than the one originally solved for; many responses said no, a more efficient allocation was impossible, by proposition 11.2-4. This answer is incorrect.

The original concept is that the unique separating equilibrium which satisfies incentive compatibility and the intuitive criterion is the Pareto efficient outcome. In particular, this outcome is known in the literature as the, "Riley outcome," in honor of Riley's original description of it in 1979 in a paper titled *Informational Equilibrium* (found in Econometrica). This concept is somewhat familiar from early in the quarter: when we analyzed Spence's educational signalling game, we whittled down the equilibrium to the unique equilibrium which satisfied the intuitive criterion. From this point, it was obvious that we could do no better for the agents as a whole without violating equilibrium restrictions.

As demonstrated in Riley's paper, this notion generalizes to a setup with continuous types. So what is violated in the homework setup? Firstly, the statement of the theorem claims only to hold with respect to the *unique* separating equilibrium satisfying the intuitive criterion; in application, we need to ensure that a particular separating equilibrium is, in fact, unique. There is a hidden assumption in *Essential Microeconomics*' treatment of the theorem (which is explicit in the original paper): utility must be differentiable with respect to the action space. In the question setup, since there are two signalling mechanisms the action space itself can be viewed as two-dimensional; naturally, there is not an immediately-convenient sense in which utility can be differentiable across the discrete jump in signalling technologies.

It's all good and well that the antecedents of the theorem don't fully obtain; how can this tell us anything further with regard to finding a more optimal outcome? Notice in the naïve "pasting" equilibrium — in which mechanisms are applied relative to a simply-calculable inflection point — there is a kink in the value function when the type crosses its critical threshold. Fundamentally, it's not unreasonable to expect that at an ideal outcome the value function will be everywhere-smooth (the first derivative should be continuous); this is because the uncovered functional form of the value function is the solution to a differential equation. Inasmuch as it is difficult to justify non-smoothness in solutions to well-behaved differential systems, we should expect that a solution to the agent's problem will be everywhere smooth.

We then look for an optimal solution which satisfies two conditions which you should see in second-year coursework: value matching and smooth pasting. Value matching says that where the two functions are glued, they must obtain the same value; this is the naïve condition we applied to obtain the first-round inflection point for switching between mechanisms. Smooth pasting says that derivatives must also be shared at the points at which the functions are glued (here, a single point). Intuitively, so long as the functions we are pasting are monotonic, the smooth pasting condition says that we can't possibly shift the "right-hand" curve upward, otherwise value matching will be violated; then since the value function cannot move any higher, it should be at the optimum achievable in incentive-compatible mechanisms.

A handful of people approached this problem using smooth pasting. They recognized that the problem was initially solved via value matching (formally or intuitively), and then looked through the two separate value functions to find the point at which the slopes were identical; this provided an offset by which to shift the right-hand value function. Unfortunately, this approach is also wrong: we cannot simply shift the value function — or approach value matching and smooth pasting separately — since it is a sensitive solution to a differential equation. The proper approach (addressed in the solutions) accounts for value matching and smooth pasting simultaneously; solving the differential equation follows the same "integrate up" principle we've been applying to many Riley problems, but we now must adjust our initial condition to respect value matching and smooth pasting. In particular, we plug in the derivative of the left-hand value function as well as its value *as the values of the differential equation expressing the right-hand value function*, which gives us the ability to solve explicitly for the point which satisfies value matching and smooth pasting. This becomes the initial condition for integration and solution of the differential equation, and from this the solution follows in the usual way. Having found the initial condition in this way, we are naturally at a value-matched and smoothly-pasted solution, which as hand-waved above should be Pareto optimal.

For more on the application of this method, I highly recommend checking the problem set solutions; they contain mathematical arguments which should make the notions more clear.

Essential Microeconomics, exercise 12.3-2

Buyer i's type θ_i is an independent draw from a distribution with support $[\alpha, \beta]$, CDF $F(\cdot)$ and density function $f(\cdot)$. His value is a symmetric function $v(\theta_1, \ldots, \theta_N)$ of each buyer's type.

(a) Argue as for the private values case that the expected surplus of buyer i is

$$E\left[\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\frac{\partial v}{\partial \theta_i}(\theta)q_i(\theta)\right]$$

Solution: naïvely, we can state the agent's value function as

$$V_i(\theta_i) = E_{\theta_{-i}}\left[q_i(\theta)v(\theta)\right] - r_i(\theta_i)$$

Since all involved functions are well-behaved, we find the first derivative of the value function to be

$$V_i'(\theta_i) = E_{\theta_{-i}}\left[q_i'(\theta)v(\theta) + q_i(\theta)\frac{\partial v}{\partial \theta_i}(\theta)\right] - r_i'(\theta)$$

Standard envelope theorem arguments may be applied to obtain neat cancellation of two of these terms; to see this, apply a direct revelation argument to see that agent i's utility from reporting type t is

$$u_i(t;\theta_i) = E_{\theta_{-i}}\left[q_i(t)v(\theta)\right] - r_i(t)$$

According to the first-order conditions for optimality, we have

$$E_{\theta_{-i}}\left[q_i'(t)v(\theta)\right] = r_i'(t)$$

For truthful reporting to be an optimum, we have this equivalence at $t = \theta_i$; that is,

$$E_{\theta_{-i}}\left[q_i'(\theta_i)v(\theta)\right] = r_i'(\theta_i)$$

From this, it's apparent that the first and last terms in the expression of $V_i'(\theta_i)$ cancel. Then we have

$$V_i'(\theta_i) = E_{\theta_{-i}}\left[q_i(\theta)\frac{\partial v}{\partial \theta_i}(\theta)\right]$$

This quantity aids in calculation of $E(V_i(\theta_i))$ in the usual way:

$$\begin{aligned}
E\left[V_i(\theta_i)\right] &= \int_\alpha^\beta V_i(\theta_i)dF(\theta_i) \\
&= -V_i(\theta_i)(1-F(\theta_i))|_{\theta_i=\alpha}^\beta + \int_\alpha^\beta V_i'(\theta_i)(1-F(\theta_i))d\theta_i \\
&= \int_\alpha^\beta V_i'(\theta_i)\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right) f(\theta_i)d\theta_i \\
&= E_{\theta_i}\left[V_i'(\theta_i)\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] \\
&= E_\theta\left[q_i(\theta)\frac{\partial v}{\partial \theta_i}(\theta)\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right]
\end{aligned}$$

Thus we obtain the desired result.

(b) Hence show that if the seller's value is zero, the expected profit of the seller is

$$E\left[U_0(\theta)\right] = E\left[\sum_{i=1}^N q_i(\theta)\left(v(\theta) - \left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\frac{\partial v}{\partial \theta_i}\right)\right]$$

Solution: by construction, we know that the seller's expected profit U_0 is the sum of expected transfers from buyers,

$$E\left[U_0(\theta)\right] = \sum_{i=1}^N E\left[r_i(\theta)\right]$$

Rearranging the value function from part (a) above, we then obtain

$$\begin{aligned}
E\left[U_0(\theta)\right] &= \sum_{i=1}^N E\left[q_i(\theta)v(\theta) - V_i(\theta)\right] \\
&= E\left[\sum_{i=1}^N q_i(\theta)v(\theta) - q_i(\theta)\frac{\partial v}{\partial \theta_i}(\theta)\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] \\
&= E\left[\sum_{i=1}^N q_i(\theta)\left(v(\theta) - \left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\frac{\partial v}{\partial \theta_i}(\theta)\right)\right]
\end{aligned}$$

(c) Hence explain why the sealed high-bid and sealed second-bid auctions are payoff-equivalent.

Solution: notice that the expressions for expected revenue and expected value depend only on the allocation function $q_i(\theta)$. That is, the remainder of the terms within the expectation are not subject to design within the mechanism. Therefore, the quantity allocation function is the unique determinant of expected revenue and value. It follows that any efficient mechanism — which allocates the good to the agent with the highest type — must have the same expected revenue, since the allocation function is identical across all such mechanisms. Since the sealed high-bid and sealed second-bid auctions are both efficient allocation mechanisms, they must then be payoff-equivalent.

(d) Suppose there are two bidders and types are uniformly distributed on $[0,1]$, and that the common value function is $v(\theta) = \sum_{i=1}^2 \theta_i$; obtain an expression for expected revenue if the seller only allocates to types above $\hat{\theta}$. Hence show that revenue is maximized when $\hat{\theta} = \frac{2}{5}$.

Solution: plugging in to the previous equation, we have

$$E[U_0] = E\left[\sum_{i=1}^{2} q_i(\theta)\left(\theta_1 + \theta_2 - \frac{1-\theta_i}{1}\right)\right] = \sum_{i=1}^{2} E\left[q_i(\theta)\left(2\theta_i + \theta_{-i} - 1\right)\right]$$

By separating the expectation into the sum over two expectations, it is clear from the symmetry in the problem that optimizing $\hat{\theta}$ with respect to a single expectation will suffice. We then check,

$$E\left[q_i(\theta)\left(2\theta_i + \theta_{-i} - 1\right)\right] = \int_{\hat{\theta}}^{1}\int_{0}^{\theta_i} 2\theta_i + \theta_{-i} - 1 d\theta_{-i} d\theta_i$$

In this expression, the allocation function $q(\cdot)$ is moved into the bounds of integration. We take the inner integral to simplify the problem, and find

$$E\left[q_i(\theta)\left(2\theta_i + \theta_{-i} - 1\right)\right] = \int_{\hat{\theta}}^{1} \frac{5}{2}\theta_i^2 - \theta_i d\theta_i$$

First-order conditions with respect to $\hat{\theta}$ give us

$$-\left(\frac{5}{2}\hat{\theta}^2 - \hat{\theta}\right) = 0$$

This is solved at $\hat{\theta} = \frac{2}{5}$. Then the revenue-maximizing threshold allocates to agents only if their type is above $\frac{2}{5}$ (and they are the high bidder).

(e) What is the revenue-maximizing reserve price in this case?

Solution: we look for a reserve price so that a bidder of type $\theta_i = \frac{2}{5}$ obtains 0 expected profit. So suppose a bidder of type $\theta_i = \frac{2}{5}$ wins; his expected surplus is

$$\begin{aligned}\int_{0}^{\frac{2}{5}} (\theta_i + \theta_{-i})\left(\frac{5}{2}\right) d\theta_{-i} &= \left(\frac{5}{2}\right)\left(\frac{2}{5}\theta_{-i} + \frac{\theta_{-i}^2}{2}\right)\Big|_{\theta_{-i}=0}^{\frac{2}{5}} \\ &= \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{5}\right)^2 \\ &= \frac{3}{5}\end{aligned}$$

Here, the $\frac{5}{2}$ term in the integral is due to the expression for the conditional density of θ_{-i}, given that it is less than θ_i. Then type $\theta_i = \frac{2}{5}$ has an expected surplus of $\frac{3}{5}$ from winning the item, and a reseve price of $\hat{b} = \frac{3}{5}$ is appropriate.

Essential Microeconomics, exercise 12.4-2

There are two agents, each with a value continuously distributed on $[0, 1]$. The cost of the public good is $k \in (1, 2)$.

(a) Show that in a VCG mechanism the equilibrium expected payoff of each agent is equal to expected total surplus.

Solution: in the VCG mechanism, the payoff to agent i of type θ_i given that all other agents are described by types θ_{-i} is

$$V_i(\theta_i|\theta_{-i}, q) = S(\theta_i, \theta_{-i}; q) - S(\alpha_i, \theta_{-i}; q)$$

where α_i is the lower bound on the support of agent i's possible types[1]. Here, $\alpha_i = 0$ for both agents. Notice that when $\theta_i = 0$, we have that $\sum_{i=1}^{2} \theta_i \leq 1 < k$, so the public good cannot be constructed; in more economic terms, this means that both agents are blocking. Then for any efficient allocation mechanism q, $S(\alpha_i, \theta_{-i}; q) = 0$.

It follows that in this VCG setup, an agent's ex post surplus is

$$V_i(\theta_i|\theta_{-i}, q) = S(\theta_i, \theta_{-i}; q)$$

In the interim phase this is

$$E_{\theta_{-i}}\left[V_i(\theta_i|\theta_{-i}, q)\right] = E_{\theta_{-i}}\left[S(\theta_i, \theta_{-i}; q)\right]$$

Ex ante, this is

$$E_\theta\left[V_i\right] = E_\theta\left[S(\theta)\right]$$

Here, we are using Riley's notation and not further parametrizing $V_i(\cdot)$, since ex ante we need only represent which agent we are obtaining the value of and have no need to indicate any notion of type in this value.

This expression is exactly what we were asked to obtain: each agent's ex ante expected utility is equal to expected total surplus.

(b) Hence explain why the expected loss of the mechanism designer is equal to the expected total surplus.

Solution: we know that each agent's value is equal to her surplus plus some transfer from the mechanism designer; ex ante, this gives us

$$E_\theta[V_i] = E_\theta[q(\theta)\theta_i + t_i(\theta)]$$

From above, we then have

$$E_\theta[S(\theta)] = E_\theta[q(\theta)\theta_i + t_i(\theta)]$$

It follows then that

$$E_\theta[t_i(\theta)] = E_\theta[S(\theta) - q(\theta)\theta_i]$$

By sign convention, $t_i(\theta) > 0$ implies a transfer to the agent from the mechanism designer, and hence a loss of $t_i(\theta)$ to the mechanism designer. Thus the designers expected total loss through transfers is

$$\sum_{i=1}^{2} E_\theta[t_i(\theta)] = E_\theta\left[\sum_{i=1}^{2} S(\theta) - \theta_i\right]$$

The definition of social surplus gives us

$$S(\theta) = q(\theta)\left(\theta_1 + \theta_2 - k\right)$$

Then the statement of the expected loss to the designer through transfers becomes

$$\begin{aligned}\sum_{i=1}^{2} E_\theta[t_i(\theta)] &= E_\theta\left[\sum_{i=1}^{2} q(\theta)\left(\theta_{-i} - k\right)\right] \\ &= E_\theta\left[q(\theta)\left(\theta_1 + \theta_2 - 2k\right)\right] \\ &= E_\theta\left[S(\theta) - q(\theta)k\right]\end{aligned}$$

[1]Intuitively, α_i represents the lowest possible belief *following any play on i's part* in a perfect Bayesian equilibrium.

Of course, if the indicated values θ are sufficiently large the mechanism designer must also fund the production of the public good. Then we introduce an additional cost $q(\theta)k$ above and beyond the specified transfer scheme; it follows that the total expected loss of the mechanism designer is

$$\sum_{i=1}^{2} E_\theta\left[t_i(\theta)\right] + E_\theta\left[q(\theta)k\right] = E_\theta\left[S(\theta)\right]$$

Then the designer's expected total loss is equal to expected total surplus.

(c) Suppose instead that there are three agents and that $k > 2$. Show again that the mechanism designer faces an expected loss. How big is this loss?

Solution: again, each agent is blocking. This is sufficient to tell us as in (a) that each agent's ex ante expected surplus is equal to expected total surplus. Appealing to the above logic, the expected transfer to any one agent i is

$$E_\theta[t_i(\theta)] = E_\theta\left[\sum_{j\neq i} q(\theta)\theta_j - q(\theta)k\right]$$

Summing total transfers, we obtain

$$\begin{aligned}
\sum_{i=1}^{3} E_\theta[t_i(\theta)] &= \sum_{i=1}^{3} E_\theta\left[\left(\sum_{j\neq i} q(\theta)\theta_j\right) - q(\theta)k\right] \\
&= 2\sum_{i=1}^{3} E_\theta\left[q(\theta)\theta_i\right] - 3E_\theta\left[q(\theta)k\right] \\
&= 2E_\theta\left[S(\theta)\right] - E_\theta[q(\theta)k]
\end{aligned}$$

Again accounting for the possibility that the public good is produced, incurring a conditional cost of k to the mechanism designer, we find that the expected total loss of the mechanism designer is the expected sum of transfers and the cost of production,

$$E_\theta\left[\sum_{i=1}^{3} t_i(\theta)\right] + E_\theta[q(\theta)k] = 2E_\theta[S(\theta)]$$

That is, the designer's expected loss is equal to twice expected total surplus. It should be evident how this situation generalizes to more agents in the economy.

Essential Microeconomics, exercise 12.5-1

There are two agents. Agent i's value of the public good is continuously distributed on the interval $\Theta_i = [\alpha_i, \beta_i]$. The cost of the public good is k.

(a) If $\Theta_i = [0, 1]$ and the cost of the public good $k \in (1, 2)$, show that for efficient public good provision the loss to the mechanism designer is at least equal to the expected social surplus.

Solution: knowledge that VCG is the efficient, incentive compatible mechanism which maximizes seller revenue is enough (together with the above question) to correctly answer this question — in fact, these two together are sufficient (at least logically) to answer all three parts. Because this method is not particularly informative, we will take a different tack here.

Suppose that agent value functions are given ex post by

$$V_i(\theta) = q(\theta)\left(\theta_i - r_i(\theta)\right)$$

From the envelope theorem, we then have (for incentive compatibility)

$$V_i'(\theta) = q(\theta)$$

Integrating up, this gives us

$$V_i(\theta) = \int_{\alpha_i}^{\theta_i} q(t, \theta_{-i})dt + V_i(0, \theta_{-i})$$

In this problem, each agent is blocking; it follows that $q(0, \theta_{-i}) = 0$ and hence $V_i(0, \theta_{-i}) = 0$. This constraint gives us a further restatement of the value function *provided that θ is large enough for production to be efficient*,

$$\begin{aligned} V_i(\theta) &= \int_{k - \sum_{j \neq i} \theta_j}^{\theta_i} dt \\ &= \theta_i - \left(k - \sum_{j \neq i} \theta_j\right) \\ &= \sum_{i=1}^{N} \theta_i - k \\ &= S(\theta, q(\theta)) \end{aligned}$$

Of course, when θ is small $V(\theta) = 0$. Notice that the derivation above depends in no way on the existence of only two agents in the economy; the crucial dependence on the problem setup is only through the fact that each agent is a blocking agent.

The designer's revenue from a particular type parametrization θ is

$$\begin{aligned} \sum_{i=1}^{N} r_i(\theta)q(\theta) &= \sum_{i=1}^{N} \theta_i q(\theta) - V_i(\theta) \\ &= S(\theta, q(\theta)) + kq(\theta) - \sum_{i=1}^{N} V_i(\theta) \\ &= S(\theta, q(\theta)) + kq(\theta) - \sum_{i=1}^{N} S(\theta, q(\theta)) \\ &= -(N-1)S(\theta, q(\theta)) + kq(\theta) \end{aligned}$$

The designer's profit is revenue less costs; we then subtract an additional $kq(\theta)$ for the possibility of the good being produced to obtain an expression for the seller's profit of

$$u_0 = -(N-1)S(\theta, q(\theta))$$

In this case, $N = 2$, so we have

$$u_0 = -S(\theta, q(\theta))$$

Then the designer's ex post profit is exactly the negative of social surplus.

(b) What if $\Theta_1 = [1, 2]$, $\Theta_2 = [2, 3]$, and $k \in (4, 5)$?

Solution: again, we have the important fact that each agent is blocking. Then none of the math above changes and the same result obtains.

(c) What if there are three agents, all with values on the interval $[0, 1]$, and $k \in (2, 3)$?

Solution: once more, each agent is blocking. The only extent to which the math above changes is in the final setting of $N = 2$; with $N = 3$ in this problem setup, we obtain

$$u_0 = -2S(\theta, q(\theta))$$

Of course, the idea of this result is not at all distinct from those previous.

Essential Microeconomics, exercise 12.5-2

Suppose that each of N agent's values is uniformly distributed on the interval $[\alpha, \beta]$, and that the cost of the public good is k. Given $\lambda \in [0, 1]$, solve for the allocation rule $q(\cdot)$ that maximizes

$$W^\lambda = (1 - \lambda)E\left[S(\theta, q(\theta))\right] + \lambda E\left[u_0\right]$$

where u_0 is the payoff to the mechanism designer.

Solution: this question asks us to maximize social welfare for varying [parametrized] definitions of social welfare; a general form for this exists in section 12.5 of *Essential Microeconomics*, but obtaining this specific result from first principles is not so terrible.

Given allocation rule $q(\cdot)$, the social surplus obtained from a particular vector of types θ is

$$S(\theta, q(\theta)) = q(\theta)\left(\sum_{i=1}^{N} \theta_i - k\right)$$

It follows that

$$E\left[S(\theta, q(\theta))\right] = E\left[q(\theta)\left(\sum_{i=1}^{N} \theta_i - k\right)\right]$$

We may express the agent's value function as

$$V_i(\theta) = q(\theta)\left(\theta_i - r_i(\theta)\right)$$

As usual, this gives us

$$r_i(\theta)q(\theta) = q(\theta)\theta_i - V_i(\theta)$$

Since the designer's expected revenue depends on whether or not the object is allocated, the expected revenue from a single agent is

$$E\left[r_i(\theta)q(\theta)\right] = E\left[q(\theta)\theta_i - V_i(\theta)\right]$$

Here, we apply our usual tricks to tease out the $E[V_i(\theta)]$ term — if these are unfamiliar, see exercise 12.3-2 above of read through the Week 4 handout — and find, by substitution

$$E\left[r_i(\theta)q(\theta)\right] = E\left[q(\theta)J_i(\theta)\right]$$

To finally determine the seller's surplus, we must add the expected revenue from each of the agents and account for the cost of provisioning the good; the seller's expected surplus is then

$$E\left[u_0\right] = E\left[\sum_{i=1}^{N} q(\theta)J_i(\theta) - q(\theta)k\right]$$

We are now set to solve the question at hand. Substituting in, we find

$$\begin{aligned}
W^\lambda &= (1-\lambda)E\left[q(\theta)\left(\sum_{i=1}^N \theta_i - k\right)\right] + \lambda E\left[q(\theta)\left(\sum_{i=1}^N J_i(\theta) - k\right)\right] \\
&= E\left[q(\theta)\left(\sum_{i=1}^N (1-\lambda)\theta_i + \lambda J_i(\theta)\right) - q(\theta)k\right] \\
&= E\left[q(\theta)\left(\sum_{i=1}^N (1-\lambda)\theta_i + \lambda\left(\theta_i - \frac{1-\frac{\theta_i-\alpha}{\beta-\alpha}}{\frac{1}{\beta-\alpha}}\right)\right) - q(\theta)k\right] \\
&= E\left[q(\theta)\left(\sum_{i=1}^N \theta_i + \lambda(\theta_i - \beta)\right) - q(\theta)k\right]
\end{aligned}$$

Expressing this in integral form, we obtain

$$\int_\alpha^\beta \cdots \int_\alpha^\beta q(\theta)\left[\left(\sum_{i=1}^N (1+\lambda)\theta_i\right) - N\lambda\beta - k\right] d\theta$$

Since $q(\cdot)$ enters nowhere into the internal term, pointwise maximization tells us that $q(\theta)$ will be 1 where the inside term is [weakly] positive, and 0 otherwise. We then solve,

$$\begin{aligned}
&\left(\sum_{i=1}^N (1+\lambda)\theta_i\right) - N\lambda\beta - k \geq 0 \\
\iff \qquad &\sum_{i=1}^N \theta_i \geq \frac{k + N\lambda\beta}{1+\lambda}
\end{aligned}$$

Then the allocation rule $q(\cdot)$ which maximizes the defined quantity W^λ is

$$q(\theta) = \begin{cases} 1 & \text{if } \sum_{i=1}^N \theta_i \geq \frac{k+N\lambda\beta}{1+\lambda} \\ 0 & \text{otherwise} \end{cases}$$

Essential Microeconomics, exercise 12.6-1

Suppose that the buyer's value is distributed on $[3, 5]$ and the seller's cost is distributed on $[4, 6]$.

(a) Convert the problem to a public goods problem and show that both agents are blocking.

Solution: we can see that the good should be produced and allocated if the seller's cost c is below the buyer's value v. This restriction, $c < v$, can be expressed as $v - c > 0$, and this in turn becomes more familiar as $v + (-c) > 0$.

The question is then describable as a public goods problem with agent 1 of type $\theta_1 \in [3, 5]$ and agent 2 of type $\theta_2 \in [-6, -4]$; provision of the public good is efficient if and only if the sum of their types is greater than 0. Note that many other normalizations could have been selected, but this seems to be the most intuitive.

Seeing that both agents are blocking in this setup is not difficult: suppose that agent 1 is of type $\theta_1 = 5$, her maximal type; then so long as $\theta_2 < -5$, it is not efficient to produce the public good. Similarly, if agent 2 is of type -4, his maximal type, then so long as $\theta_1 < 4$ it is not efficient to produce the public good. This is the definition of the agents being blocking.

(b) If both cost and value are uniformly distributed, what is the minimized expected loss of the designer in an efficient exchange mechanism?

Solution: we know that the VCG mechanism maximizes designer profit, which is a mere sign convention away from minimizing loss. So we construct a VCG mechanism here and examine the designer's expected loss.

In VCG, each agent's payoff is

$$u_i(\theta_i, \theta_{-i}; q) = S(\theta; q) - S(\alpha_i, \theta_{-i}; q)$$

where α_i is the lower bound of the agent's type support. Having already seen that both agents are blocking, we know that social surplus when either agent reports their lowest type is 0, and hence $S(\alpha_i, \theta_{-i}; q) = 0)$ in any efficient allocation rule. It follows that

$$u_i(\theta_i, \theta_{-i}; q) = S(\theta; q) = q(\theta)(\theta_1 + \theta_2)$$

We may separately state the agents' utility functions as their surplus from allocation plus a transfer from the mechanism designer,

$$u_i(\theta_i, \theta_{-i}; q) = q(\theta)\theta_i + t_i(\theta)$$

Equating this equation and the previous, we obtain

$$t_i(\theta) = q(\theta)\theta_{-i}$$

Summing over total transfers, the payouts from the designer to the agents are

$$\sum_{i=1}^{2} t_i(\theta) = q(\theta)\sum_{i=1}^{2}\theta_{-i} = S(\theta; q)$$

Since in our public goods restatement of the problem production of the good costs 0, the designer faces no additional cost whether or not the good is produced. Hence the total loss to the designer remains $S(\theta; q)$; this is precisely the form we've obtained in several previous questions where agents are blocking.

With the designer's loss equal to total surplus, all that remains is to calculate expected total surplus. We find

$$\begin{aligned}
E_\theta[S(\theta; q)] &= \int_3^5 \int_{-6}^{-4} q(\theta)(\theta_1 + \theta_2)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) d\theta_2 d\theta_1 \\
&= \frac{1}{4}\int_4^5 \int_{-\theta_1}^{-4} \theta_1 + \theta_2 d\theta_2 d\theta_1 \\
&= \frac{1}{4}\int_4^5 \theta_1\theta_2 + \frac{\theta_2^2}{2}\bigg|_{\theta_2=-\theta_1}^{-4} d\theta_1 \\
&= \frac{1}{4}\int_4^5 \frac{1}{2}\theta_1^2 - 4\theta_1 + 8 d\theta_1 \\
&= \frac{1}{4}\left(\frac{\theta_1^3}{6} - 2\theta_1^2 + 8\theta_1\right)\bigg|_{\theta_1=4}^{5} \\
&= \frac{1}{4}\left(\frac{125}{6} - 50 + 40\right) - \frac{1}{4}\left(\frac{64}{6} - 32 + 32\right) \\
&= \frac{1}{24}
\end{aligned}$$

Thus the designer's expected loss is $\frac{1}{24}$.

2007 Fall comp, question 4

The government of Ketchikan, Alaska, is contemplating building a footbridge to a small island that has only three residents. The bridge would cost 30 (thousands of dollars), and the cost must be shared by the residents; that is, the government is not allowed to subsidize construction or to have any money left over. Each of the residents knows her own valuation for use of the bridge, but not the valuation of the other residents; the government does not know the valuation of any resident. The government and the residents each think the private valuations are either High or Low with equal and independent probability; for residents 1 and 2, the High valuation is 10 and the Low valuation is 0; for resident 3 the High valuation is V and the Low valuation is 0. (V is a parameter of the problem).

For what values of the parameter V can the government find an efficient, individually-rational, incentive-compatible mechanism? You may assume that the mechanism treats residents 1 and 2 symmetrically.

Solution: immediately, we can see that VCG will not be an appropriate mechanism in this case; by construction, VCG offers a set of transfers from the mechanism designer to the agents in the system, and this set of transfers is not constrained by budget balance. Since in this question the bridge must be fully-funded by the population of the island with no financial intervention from any government entity, such a set of transfers is not feasible. We are then forced to look for alternate mechanisms.

Key in this question is that V is a parameter to the model. That is, agent 3 only needs to be incentivized to indicate V versus 0 (or vice-versa) and does not need to be incentivized to indicate a particular value of V. We can then solve this question for particular values of V; we will assume $V > 0$, although assuming otherwise has no practical effects (the model itself is slightly perverse in setup, though).

Case 1: $V < 10$. Then even if all agents are of the high type, $10 + 10 + V < 30$ and building the bridge is *never* efficient. An efficient, incentive-compatible mechanism will then charge the agents nothing and not build the bridge regardless of their reports.

Case 2: $V \geq 30$. Here, agent 3 alone may value the bridge Highly and construction is still efficient; further, without agent 3's approval the bridge cannot be built. An efficient, incentive-compatible mechanism with budget balance may then build the bridge if and only if agent 3 reveals a High type; this mechanism can, for example, bill the entirety of construction costs to agent 3 (although there is a continuum of other options for allocating the cost across agents; this mechanism is merely the simplest).

Case 3: $10 \leq V < 20$. This is the case in which building the bridge *may* be efficient, but all three agents must value the bridge Highly in order for it to be constructed. An efficient mechanism will then build the bridge if and only if all three agents report a High type; one method of ensuring incentive compatibility is to require them to split the cost equally, each paying 10 (although again there is a continuum of other payment schemes).

Case 4: $20 \leq V < 30$. As always, efficiency of construction implies that the bridge should only be built if agent 3 reports a High type; however, in this case only one other agent is needed for the social value of the bridge to exceed the cost of construction. We find that here, incentivizing agents to report truthfully is slightly more difficult, as agents 1 and 2 may now hope that their counterpart reveals a High value and accepts the cost of construction, leaving them the freedom to misreport and obtain the entirety of their surplus with no payment; that is, this is the unique case in which the free rider problem truly shows.

Suppose that we have a mechanism in which agents 1 and 2 pay amount p_n if the bridge is built and they (individually), along with n other agents have revealed a High type (that is, if 1 and 3 reveal a High type while 2 reveals a Low type, 1's payment will be p_1). Assume a similar notation for agent 3, with p_n^V being the amount which she pays if she is of the high type and n other agents are, as well.

For incentive compatibility for the lower agents, we have

$$E\left[u_i(\text{Low};\text{High})\right] \le E\left[u_i(\text{High};\text{High})\right]$$

More formally, this yields

$$\begin{aligned}
& 10\Pr(\theta_2=\text{H}\wedge\theta_3=\text{H}) \le (10-p_1)\Pr(\theta_2=\text{L}\wedge\theta_3=\text{H}) + (10-p_2)\Pr(\theta_2=\text{H}\wedge\theta_3=\text{H}) \\
\iff\quad & 10\left(\frac{1}{4}\right) \le (10-p_1)\left(\frac{1}{4}\right) + (10-p_2)\left(\frac{1}{4}\right) \\
\iff\quad & p_1+p_2 \le 10
\end{aligned}$$

A similar condition for agent 3 will yield

$$\begin{aligned}
& 0 \le (V-p_1^V)\Pr(\theta_1=\text{H}\wedge\theta_2=\text{L}) + (V-p_1^V)\Pr(\theta_1=\text{L}\wedge\theta_2=\text{H}) + (V-p_2^V)\Pr(\theta_1=\text{H}\wedge\theta_2=\text{H}) \\
\iff\quad & 0 \le (V-p_1^V)\left(\frac{1}{2}\right) + (V-p_2^V)\left(\frac{1}{4}\right) \\
\iff\quad & 3V \ge 2p_1^V + p_2^V
\end{aligned}$$

Budget balance — the precise clearing of the funding for bridge construction — tells us that $p_1^V = 30-p_1$ and $p_2^V = 30-2p_2$. Then the above inequality becomes

$$p_1+p_2 \ge \frac{90-3V}{2}$$

We then have a range of consistent solutions for the quantity p_1+p_2,

$$\frac{90-3V}{2} \le p_1+p_2 \le 10$$

Solving out, a consistent solution will then exist only when

$$V \ge \frac{70}{3}$$

That is, for $V \in [20,30)$ as long as $V \ge \frac{70}{3}$ we can generate a symmetric payment scheme which is incentive compatible and efficiently funds bridge construction with budget balance.

From the above cases, we see that we can form an incentive compatible, individually rational, efficient mechanism for financing bridge construction if and only if

$$V \in (0,20) \cup \left[\frac{70}{3},+\infty\right)$$

2008 Fall comp, question 4

Two bidders compete in a closed-bid auction for a single, indivisible item. The high bidder wins the item and pays the amount of his bid. Bidders' signals are drawn independently from the uniform distribution on $[0,1]$; if signals are s_1, s_2 then the common value of the object is $s_1 s_2$.

Find the unique symmetric equilibrium in smooth, strictly-increasing bid functions[2].

[2] Note that we *did not* work with common-value auctions this year; however, since we have the tools available to solve these questions, it's a useful check of our ability to apply our knowledge to variants of recognized setups.

Solution: we have the capacity to solve this exactly as in exercise 12.3-2 from *Essential Microeconomics*, as above. However, this question is just as well modeled as a Bayesian game. Suppose an agent receives signal s_i; his expected utility from playing as if he received signal t is

$$u_i(t; s_i) = E_{s_{-i}}\left[q_i(t, s_{-i})\left(s_i s_{-i} - b(t)\right)\right]$$

Here, $q_i(t, s_{-i})$ is the quantity allocated to agent i from playing according to signal t when agent $-i$ plays according to signal s_{-i}.

Since we are analyzing a first-price auction, it is immediate that

$$q_i(t, s_{-i}) = \chi(b(t) > b(s_{-i}))$$

That is, if both agents play according to the same bid strategy $b(\cdot)$ — which we are given by our assumption of symmetry — the agent receives the entirety of the item if and only if his bid is higher than agent $-i$'s.

In our standard setup of private values, we would immediately take first-order conditions of the agent's utility function. However, in the common values setup, we cannot completely separate the win probability/allocation function from our value of the item; that is, if agent i wins the item he knows that he values the item more highly (in expectation) than agent $-i$. Since values are common, this roughly implies that he may have overbid for the object, a phenomenon known as the winner's curse. This conditional information structure is encapsulated in the expression of expectation, but does not appear when we separate out the win probability from the valuation as we do in the private values case. Of course, as we saw in exercise 12.3-2 above, these terms tend to cancel anyway, but we need to watch our intuition before blazing through these problems.

According to the uniform distribution on $[0, 1]$, we can express the above expectation as

$$\begin{aligned} E_{s_{-i}}\left[q_i(t, s_{-i})\left(s_i s_{-i} - b(t)\right)\right] &= \int_0^1 \chi(b(t) > b(s_{-i}))\left(s_i s_{-i} - b(t)\right) ds_{-i} \\ &= \int_0^t s_i s_{-i} - b(t) ds_{-i} \end{aligned}$$

That is, when bid functions are strictly increasing having a higher bid is identical to receiving a higher signal, so we can alter the bounds of the integral and eliminate the indicator function.

To optimize, we take first-order conditions with respect to the agent's indication t. We have the option of expressing the integral quantity precisely prior to doing so, but I am a fan of the fundamental theorem of calculus; we then obtain

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= s_i t - b(t) - \int_0^t b'(t) ds_{-i} \\ &= s_i t - b(t) - t b'(t) \end{aligned}$$

To ensure truthful reporting at the optimum, we need $t = s_i$; so we obtain

$$b(s_i) + s_i b'(s_i) = s_i^2$$

From our standard box of tricks, this is representable as

$$\frac{\partial}{\partial s_i}\left[s_i b(s_i)\right] = s_i^2$$

Integrating up and solving out, we obtain

$$b(s_i) = \frac{1}{3} s_i^2$$

This is the unique symmetric, smooth, strictly-increasing bid function. This result may be just as well obtained from a mechanism design perspective as in exercise 12.3-2; as a quick exercise, make this attempt and validate our result here.

2009 Fall comp, question 4

In 2010, Christian Hellwig spent a fair amount of time covering what are referred to as, "jury voting problems." Although we have not seen them this year, they provide decent perfect Bayesian equilibrium practice.

Three jurors are deciding the fate of a person charged with murder. The person could be innocent or guilty, $\omega \in \{I, G\}$, with a prior $\Pr(\omega = G) = \mu \in (1-p, \frac{1}{2})$. Jurors A and B are known to be open-minded and each have an assessment (or signal) of the defendant's guilt, $x_i \in \{I, G\}$, with $\Pr(x_i = \omega) = p > \frac{1}{2}$, for $i \in \{A, B\}$. Juror C, on the other hand, has slept through the entire trial proceedings and will randomize, convicting with probability $\frac{1}{2}$. The jurors equally dislike convicting an innocent and not convicting a guilty person; they obtain no utility from justice being properly served.

(a) Suppose first that the jury convicts only if there is unanimous agreement to convict. Show that there exists an equilibrium in which both open-minded jurors vote according to their signals.

Solution: key in jury voting problems is the notion that an agent's decision only matters if she is "pivotal;" that is, we perform the entire equilibrium analysis under the assumption that the outcome of the jury's voting process depends only on voter i's decision. In the context of a unanimity rule, this is equivalent to the juror knowing that the other two jurors have voted to convict.

Of course, there is some information content in this situation; although one of the jurors is voting arbitrarily, the other is (by assumption from the question statement) truthfully voting according to his signal. Then if agent i is pivotal, it must be that both other jurors have voted to convict, and hence the juror who is voting truthfully must have received a signal of Guilty.

We can now analyze the payoff of agent i. Suppose she has received a signal of Guilty; if she votes to convict, her payoff is

$$u_i(\mathrm{C}|\mathrm{G}) = \Pr(\mathrm{G}|s_i = \mathrm{G} \wedge \text{pivotal})(0) + \Pr(\mathrm{I}|s_i = \mathrm{G} \wedge \text{pivotal})(-1)$$

Here, the payoff of 0 from justified conviction follows from the question statement; the payoff of -1 from unjust conviction is a normalization of the statement that the juror values being incorrect equally whether it is a false conviction or a false acquittal.

Following Bayes' rule, we have

$$\begin{aligned}
\Pr(\mathrm{I}|s_i = \mathrm{G} \wedge \text{pivotal}) &= \frac{\Pr(\mathrm{I} \wedge s_i = \mathrm{G} \wedge \text{pivotal})}{\Pr(s_i = \mathrm{G} \wedge \text{pivotal})} \\
&= \frac{\Pr(\mathrm{I} \wedge s_i = \mathrm{G} \wedge s_{-i} = \mathrm{G})}{\Pr(\mathrm{I} \wedge s_i = \mathrm{G} \wedge \text{pivotal}) + \Pr(\mathrm{G} \wedge s_i = \mathrm{G} \wedge \text{pivotal})} \\
&= \frac{(1-\mu)(1-p)(1-p)}{(1-\mu)(1-p)(1-p) + \mu p^2} \\
&= \frac{(1-\mu)(1-p)^2}{(1-\mu)(1-p)^2 + \mu p^2}
\end{aligned}$$

Then the expected utility from voting to convict when the signal is guilty, given the unanimity rule and that the juror is pivotal is

$$u_i(\mathrm{C}|\mathrm{G}) = -\frac{(1-\mu)(1-p)^2}{(1-\mu)(1-p)^2 + \mu p^2}$$

The expected utility from voting to acquit when the signal is guilty will follow a similar algebraic derivation, with a different numerator of $\Pr(\mathrm{G} \wedge s_i = \mathrm{G} \wedge \text{pivotal})$. Then we obtain an expression for this quantity of

$$u_i(\mathrm{A}|\mathrm{G}) = -\frac{\mu p^2}{(1-\mu)(1-p)^2 + \mu p^2}$$

Is truthful reporting conditional on witnessing a Guilty signal optimal? We compare:

$$
\begin{aligned}
& -\frac{(1-\mu)(1-p)^2}{(1-\mu)(1-p)^2+\mu p^2} > -\frac{\mu p^2}{(1-\mu)(1-p)^2+\mu p^2} \\
\iff \quad & (1-\mu)(1-p)^2 < \mu p^2 \\
\iff \quad & \frac{1-\mu}{\mu} < \left(\frac{p}{1-p}\right)^2
\end{aligned}
$$

By definition, $\mu \in (1-p, \frac{1}{2})$; it follows that

$$\frac{1-\mu}{\mu} < \frac{1-\mu}{1-p} < \frac{p}{1-p}$$

Since $p > \frac{1}{2}$, we also know

$$\frac{p}{1-p} < \left(\frac{p}{1-p}\right)^2$$

It follows that truthful reporting *conditional on witnessing a signal of Guilty* is an equilibrium strategy.

We now check that the same relation holds for truthful reporting conditional on witnessing a signal of Innocent. As above, the expected utility from acquitting given a signal of Innocent is

$$u_i(\mathrm{A}|\mathrm{I}) = \Pr(\mathrm{I}|s_i = \mathrm{I} \wedge \text{pivotal})(0) + \Pr(\mathrm{G}|s_i = \mathrm{I} \wedge \text{pivotal})(-1)$$

Again following Bayes' rule, we have

$$
\begin{aligned}
\Pr(\mathrm{G}|s_i = \mathrm{I} \wedge \text{pivotal}) &= \frac{\Pr(\mathrm{G} \wedge s_i = \mathrm{I} \wedge s_{-i} = \mathrm{G})}{\Pr(\mathrm{G} \wedge s_i = \mathrm{I} \wedge s_{-i} = G) + \Pr(\mathrm{I} \wedge s_i = \mathrm{I} \wedge s_{-i} = G)} \\
&= \frac{\mu(1-p)p}{\mu(1-p)p + (1-\mu)p(1-p)} \\
&= \mu
\end{aligned}
$$

Similar logic will give us that

$$\Pr(\mathrm{I}|s_i = \mathrm{I} \wedge \text{pivotal}) = 1-\mu$$

Truthful reporting conditional on receiving a signal of Innocent is then optimal if and only if

$$
\begin{aligned}
& -\mu > -(1-\mu) \\
\iff \quad & 1-\mu > \mu
\end{aligned}
$$

But since $\mu \in (1-p, \frac{1}{2})$, we know $1-\mu > \mu$. It follows that agent i's expected utility from voting according to her signal is greater than that from voting opposite her signal; thus truthful reporting is optimal conditional on receiving a signal of Innocent.

Since truthful reporting is optimal regardless of which signal is received, we see that in the mechanism with unanimous consent for conviction, truthful reporting is a Bayesian Nash equilibrium.

(b) Suppose now that the jury decides according to a simple majority rule, and requires the agreement of two jurors to convict.

Does there exist a sincere voting equilibrium in which both open-minded jurors vote according to their signals?

Solution: the situation is now complicated from the above; if a juror is pivotal in this setup, she has no way of knowing if her fellow juror who is paying attention voted to convict, or if the nonchalant juror voted to convict. Intuitively, since the prior indicates that Innocence is more likely, we may see that voting to convict when a signal of Guilty is witnessed yields a lower utility than voting to acquit.

As above, conditional on a Guilty signal being witnessed, voting according to the signal requires (assuming a truthful report on the behalf of the other agent),

$$\begin{aligned}
& E[u_i(\mathrm{C}|\mathrm{G})] \geq E[u_i(\mathrm{A}|\mathrm{G})] \\
\iff\quad & -\Pr(\mathrm{I}|s_i = \mathrm{G} \wedge \text{pivotal}) \geq -\Pr(\mathrm{G}|s_i = \mathrm{G} \wedge \text{pivotal}) \\
\iff\quad & \frac{\Pr(\mathrm{I} \wedge s_i = \mathrm{G} \wedge \text{pivotal})}{\Pr(s_i = \mathrm{G} \wedge \text{pivotal})} \leq \frac{\Pr(\mathrm{G} \wedge s_i = \mathrm{G} \wedge \text{pivotal})}{\Pr(s_i = \mathrm{G} \wedge \text{pivotal})} \\
\iff\quad & \Pr(\mathrm{I} \wedge s_i = \mathrm{G} \wedge \text{pivotal}) \leq \Pr(\mathrm{G} \wedge s_i = \mathrm{G} \wedge \text{pivotal}) \\
\iff\quad & \Pr(\mathrm{I} \wedge s_i = \mathrm{G} \wedge s_{-i} = \mathrm{G} \wedge a_n = \mathrm{A}) \qquad \Pr(\mathrm{G} \wedge s_i = \mathrm{G} \wedge s_{-i} = \mathrm{G} \wedge a_n = \mathrm{A}) \\
& + \Pr(\mathrm{I} \wedge s_i = \mathrm{G} \wedge s_{-i} = \mathrm{I} \wedge a_n = \mathrm{C}) \leq + \Pr(\mathrm{G} \wedge s_i = \mathrm{G} \wedge s_{-i} = \mathrm{I} \wedge a_n = \mathrm{C}) \\
\iff\quad & (1-\mu)(1-p)^2\left(\frac{1}{2}\right) + (1-\mu)(1-p)p\left(\frac{1}{2}\right) \leq \mu p^2\left(\frac{1}{2}\right) + \mu p(1-p)\left(\frac{1}{2}\right) \\
\iff\quad & (1-\mu)(1-p) \leq \mu p \\
\iff\quad & 1 - \mu - p \leq 0 \\
\iff\quad & \mu \geq 1 - p
\end{aligned}$$

Of course, since $\mu \in (1-p, \frac{1}{2})$ by assumption, this obtains. Then voting according to a received signal of Guilty is optimal in equilibrium.

We can perform an analogous calculation conditional on receiving a signal of Innocent,

$$\begin{aligned}
& E[u_i(\mathrm{A}|\mathrm{I})] \geq E[u_i(\mathrm{C}|\mathrm{I})] \\
\iff\quad & -\Pr(\mathrm{G}|s_i = \mathrm{I} \wedge \text{pivotal}) \geq -\Pr(\mathrm{I}|s_i = \mathrm{I} \wedge \text{pivotal}) \\
\iff\quad & \frac{\Pr(\mathrm{G} \wedge s_i = \mathrm{I} \wedge \text{pivotal})}{\Pr(s_i = \mathrm{I} \wedge \text{pivotal})} \leq \frac{\Pr(\mathrm{I} \wedge s_i = \mathrm{I} \wedge \text{pivotal})}{\Pr(s_i = \mathrm{I} \wedge \text{pivotal})} \\
\iff\quad & \Pr(\mathrm{G} \wedge s_i = \mathrm{I} \wedge \text{pivotal}) \leq \Pr(\mathrm{I} \wedge s_i = \mathrm{I} \wedge \text{pivotal}) \\
\iff\quad & \Pr(\mathrm{G} \wedge s_i = \mathrm{I} \wedge s_{-i} = \mathrm{G} \wedge a_n = \mathrm{A}) \qquad \Pr(\mathrm{I} \wedge s_i = \mathrm{I} \wedge s_{-i} = \mathrm{G} \wedge a_n = \mathrm{A}) \\
& + \Pr(\mathrm{G} \wedge s_i = \mathrm{I} \wedge s_{-i} = \mathrm{I} \wedge a_n = \mathrm{C}) \leq + \Pr(\mathrm{I} \wedge s_i = \mathrm{I} \wedge s_{-i} = \mathrm{I} \wedge a_n = \mathrm{C}) \\
\iff\quad & \mu(1-p)p\left(\frac{1}{2}\right) + \mu(1-p)^2\left(\frac{1}{2}\right) \leq (1-\mu)p(1-p)\left(\frac{1}{2}\right) + (1-\mu)p^2\left(\frac{1}{2}\right) \\
\iff\quad & \mu(1-p) \leq (1-\mu)p \\
\iff\quad & \mu \leq p
\end{aligned}$$

Again, this follows by the assumption that $p > \frac{1}{2}$ and $\mu \in (1-p, \frac{1}{2})$. Hence voting according to the signal is optimal upon receipt of Innocent.

It follows that sincere voting is supportable as an equilibrium with a majority criterion.

(c) Characterize an equilibrium of the voting game under simple majority rule. How does it compare to the equilibrium where all 3 jurors are open-minded?

Solution: this follows from the above[3]. When all three jurors are open-minded, we can support sincere voting following the exact same mathematical derivations.

(d) Are the equilibria characterized under (a) and (c) unique, or do there exist other equilibria in the jury voting game?

[3] It's unclear what the question is getting at here; is this a double-check of part (b)? That is, if you had an algebra error above, you might still get an equilibrium here.

Solution: in (a), we can certainly support an additional equilibrium of always acquit; if the informed jurors are always acquitting, no juror alone has the capacity to change his payoff. Hence always acquitting supports itself as an equilibrium in a unanimous decision context.

Can this be supported in a majority-voting context? We need to check that agents would still be willing to vote to acquit even if they witness a Guilty signal. By the setup from parts (a) and (b), being willing to acquit conditional on witnessing a Guilty signal is equivalent to

$$\begin{aligned} & -\Pr(\mathrm{I} \wedge s_i = \mathrm{G} \wedge a_n = \mathrm{C}) \leq -\Pr(\mathrm{G} \wedge s_i = \mathrm{G} \wedge a_n = \mathrm{C}) \\ \iff \quad & \Pr(\mathrm{I} \wedge s_i = \mathrm{G} \wedge a_n = \mathrm{C}) \geq \Pr(\mathrm{G} \wedge s_i = \mathrm{G} \wedge a_n = \mathrm{C}) \\ \iff \quad & (1-\mu)(1-p)\left(\frac{1}{2}\right) \geq \mu p \left(\frac{1}{2}\right) \\ \iff \quad & 1 - \mu - p \geq 0 \end{aligned}$$

We have already seen that $1-\mu-p<0$, so it follows that always voting to acquit is not an equilibrium strategy. Similar logic will show us that always voting to convict is not an equilibrium strategy (both of these results follow from the intuitive concept that if a particular strategy is followed by the other juror for sure, regardless of signal, there is no longer any information content in being pivotal).

We might be able to further construct an equilibrium in which agents vote in opposition to their signal, but that seems to be beyond the point.

If you find an algebra mistake in these derivations, please let me know; the answer seems to be in opposition to the implicit direction of the question, but the math appears to check out.

2010 Fall comp, question 4

There are two buyers. Buyer i's valuation is an independent draw from a distribution with support $[0,1]$ and CDF $F(v) \in C^1$. The seller announces a direct revelation mechanism with symmetric equilibrium allocation rule and expected payment $(\pi(v), r(v))$, where $\pi(v)$ is the probability a buyer is assigned the item if the value he submits is higher than the value announced by his opponent. The buyer who submits the lower value wins the item with zero probability. (In the case of a tie, it does not matter who is assigned the item since, in equilibrium, this will happen with zero probability.)

(a) Explain briefly why $\pi(v)$ must be non-decreasing.

Solution: the simplest statement is that, if $\pi(v)$ is decreasing, higher-typed agents may be incentivized to report a lower valuation and hence reduce the seller's expected revenue. Of course, the word "may" here is loaded with assumptions on the expected payment function (among other things) but, intuitively, if an agent of type v is to be allocated the good with a higher probability than an agent of type $u > v$, both agents' incentive constraints cannot be valid. In a way, this is a statement of single-crossing.

(b) Show that the equilibrium marginal informational rent is

$$\frac{\partial U_i}{\partial v_i} = \pi(v_i)F(v_i) \tag{1}$$

Solution: we did not cover this particular concept this year, but Riley defines the marginal informational rent (in *Essential Microeconomics*) roughly as the first derivative of the value function. Here, the value function is

$$V_i(v) = \pi(v)F(v)v - r(v)$$

Following our usual envelope theorem arguments, we have

$$V_i'(v) = \pi(v)F(v)$$

Using the suggestive notation above, in an incentive-compatible mechanism an agent's utility is given by $U_i = V_i(v)$. Hence we can describe the marginal informational rent by

$$\frac{\partial U_i}{\partial v} = \pi(v)F(v)$$

(c) For the special cases of the standard sealed, high-bid and sealed, second-bid auctions explain why (1) implies that the equilibrium buyer payoffs are the same in the two auctions.

Solution:

(d) Does revenue equivalence also follow?

Solution: yes. Since each of these mechanisms is efficient, expected buyer surplus is identical across the two mechanisms. As the seller's revenue is the residual of surplus less agent payoffs, since the agents' expected payoffs are identical in the two mechanisms it follows that the seller's revenue must also be identical.

(e) Explain why the expected revenue of the seller is

$$\overline{U}_0 = 2\int_0^1 F(v)\pi(v)vF'(v)dv - \overline{U}_1 - \overline{U}_2$$

Solution: as above, the expected revenue of the seller is the expected surplus of the buyers less the value each buyer receives (in expectation). Then we know

$$\begin{aligned}\overline{U}_0 &= E\left[F(v)\pi(v)v\right] - E\left[V_1(v)\right] + E\left[F(v)\pi(v)v\right] - E\left[V_2(v)\right] \\ &= 2\int_0^1 F(v)\pi(v)v dF(v) - \overline{U}_1 - \overline{U}_2\end{aligned}$$

(f) For the uniform case, write down an expression for expected revenue and hence solve for the $\pi(\cdot)$ which maximizes the payoff to the seller.

Solution: we may more-explicitly write

$$\begin{aligned}\overline{U}_i &= \int_0^1 V_i(v)dF(v) \\ &= \int_0^1 V_i'(v)(1-F(v))dv \\ &= \int_0^1 F(v)\pi(v)\left(\frac{1-F(v)}{F'(v)}\right)dF(v)\end{aligned}$$

This follows from our usual tricks.

Then the expression for seller revenue becomes

$$\overline{U}_0 = 2\int_0^1 F(v)\pi(v)\left(v - \left(\frac{1-F(v)}{F'(v)}\right)\right)dF(v)$$

In the uniform case, this may be rewritten as

$$\overline{U}_0 = 2\int_0^1 v\pi(v)\left(v - \left(\frac{1-v}{1}\right)\right)dF(v) = 2\int_0^1 v\pi(v)\left(2v-1\right)dF(v)$$

This integral is maximized by $\pi(v)$ such that

$$\pi(v) = \begin{cases} 1 & \text{if } v \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

(g) How might this direct mechanism be implemented as an auction?

Solution: this mechanism may be implemented as an auction with a reserve price. If we can induce agents to participate if and only if their valuation is above $\frac{1}{2}$, we will have obtained the desired mechanism. Of course, in this case, we can set a reserve price of $\hat{v} = \frac{1}{2}$, and allocate the good to the agent who bids the highest (above the reservation price).