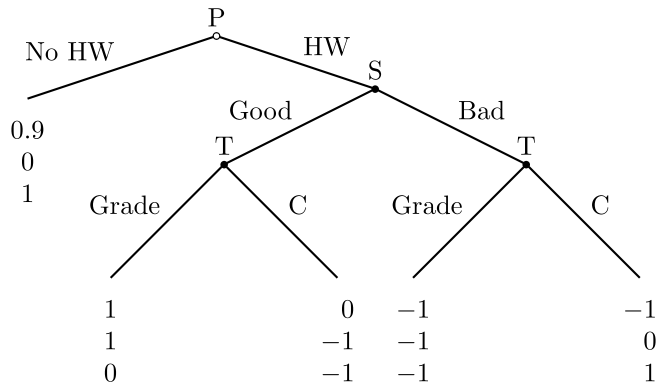
The homework game



In the homework game, a professor must choose whether or not to assign homework to a class; if the professor assigns homework, the student must choose whether to put forth good effort or bad effort, and the TA must choose whether to grade the problem set or simply assign the student a C. The payoffs from various behaviour may be justified under the following assumptions: the professor and the TA are both lazy and would rather not do work, however each feels some satisfaction when the student does well in the class. The student feels a sense of accomplishment from having her effort rewarded; if the student gets a C which she does not deserve — let's suppose she can do better if she chooses to put forth good effort — then she complains to the professor who yells at the TA, and no one in the game is happy.

How should we look for equilibrium behaviour in this game? We could attempt to translate this extensive form into a normal form and then look for Nash equilibrium, but in general when we're given an extensive-form game the simplest method of analysis will be to use backwards induction to uncover the subgame perfect Nash equilibria of the game.

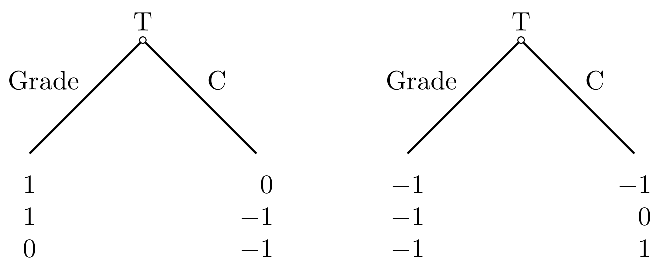


Figure 1: the two subgames of the TA in the homework game.

To do this, we begin by solving for the TA's choice at each of his subgames; these can be seen in figure 1. In the left-hand subgame, the TA prefers to grade (and receive 0 versus −1 for assigning a C); in the right-hand subgame, the TA prefers to assign a C (and receive 1 versus −1 for grading).

With this in hand, we are prepared to analyse the student's subgame; this can be seen in figure 2. Here, the student knows that if she puts forth good effort, the TA will grade her homework, while if she puts forth bad effort the TA will not grade her homework. Then she knows that she will receive a payoff of 1 for putting forth good effort and a payoff of 0 for putting forth bad effort; she then strictly prefers putting forth good effort to bad, and this is her subgame-perfect strategy.

To finish analysis, we apply these results to the professor's subgame; this can be seen in figure 3. If the professor assigns homework, the student will put forth good effort and the TA will grade the homework and

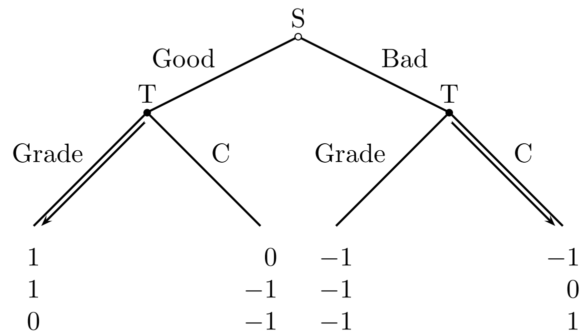


Figure 2: the student's subgame in the homework game, understanding the TA's behaviour in the next round.

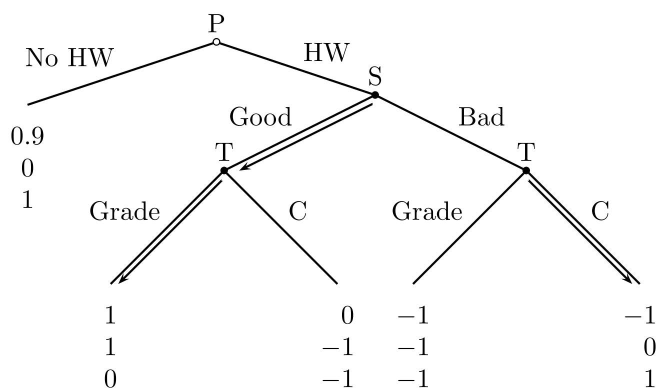


Figure 3: the professor's subgame, understanding the student's and TA's behaviour in the subsequent rounds.

the professor will receive a payoff of 1, while if he does not assign homework the game ends immediately and he will receive a payoff of 0.9. Then the professor will assign homework in a subgame-perfect Nash equilibrium.

Since we have now backed all the way through the game tree, we see that the unique subgame-perfect Nash equilibrium in this game is $(HW, G, (G|G, C|B))$ where G means "Good" for the student and "Grade" for the TA.

However, this model isn't entirely accurate; intuitively, part of the reason TAs *do* grade problem sets is to check whether or not the student has put forth effort (and is understanding the material). Speaking as a TA, I can say that we do not generally know the quality of effort put forth before the homework has been graded. Then it is a little unreasonable, perhaps, to assume that the TA can view the student's choice of effort before deciding whether or not to grade the problem set. We address this complication by introducing an information set to the extensive-form game, to indicate that the TA does not know how much effort has been put into the homework. This is represented in figure 4

How should we analyse this game? Since there is still some notion of subsequent play — the professor moves first, followed by the student and TA moving simultaneously — it is natural to apply subgame perfection. We run into trouble, though: what is the "lowest" subgame in this game? Since a subgame must begin with

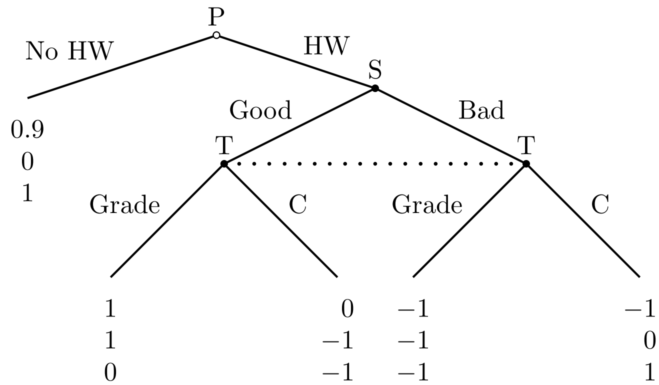


Figure 4: the extensive-form homework game, modified so that the TA cannot observe the student's effort prior to deciding whether or not to grade.

a node which is the only node in its information set, the TA has no proper subgames in this game (he cannot tell the difference between the two nodes at which he chooses an action). Then we look back a level, and see that the "lowest" subgame in this game is the student's subgame.

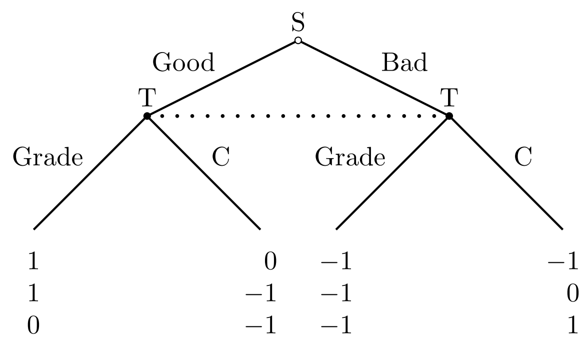


Figure 5: the student's subgame in the homework game where the TA cannot observe the student's effort choice.

The student's subgame is represented in figure 5. What constitutes Nash equilibrium in this subgame? To find Nash equilibrium, it will help to return this subgame to an equivalent normal form. Since actions are not observable, this translation is fairly straightforward and we do not need to consider contingent strategies.

Notice that in this translation, we have dropped the professor's payoff from the game form; since the professor's payoff affects neither the student's payoff nor the TA's payoff, there is no reason to include it in the analysis of this subgame. We will bring it back later when we need to.

We can find Nash equilibrium in the normal form of this subgame in the way we always do: underlining best-response strategies and looking for cells (pairs of strategies) which are mutual best responses; this is done in figure 7. We can then see that there are two pure-strategy Nash equilibria of this game: (G, G) and (B, C) (where the abbreviations are via the first letter of the strategy they represent).

To find the subgame-perfect Nash equilibrium, we then look at the professor's problem assuming Nash

	Grade	C
Good	$1, 0$	$-1, -1$
Bad	$-1, -1$	$0, 1$

Figure 6: the student's subgame in the homework game where the TA cannot observe the student's effort choice, represented as a normal-form game.

	Grade	C
Good	$\underline{1}, \underline{0}$	$-1, -1$
Bad	$-1, -1$	$\underline{0}, \underline{1}$

Figure 7: underlining best-response strategies.

equilibrium is played in subsequent stages. However, this is no longer as simple as it was: previously, there was a unique subgame-perfect Nash equilibrium at the student's subgame, but we now have two subgame-perfect Nash equilibria to concern ourselves with. The professor's choice of action in the first stage may then depend on which equilibrium is being played further down the game tree. To account for this, we base the professor's decision on what the student and TA are doing later.

- *The student and TA play* (G, G). If the student and the TA play (G, G) in the second-stage subgame, the professor's choice is identical to the previous analysis (where the TA could observe the student's choice of effort); although the information structure of the game is different, since the subsequent strategies are identical the professor's view of the game will not change. Then as before, he sees that he can obtain a payoff of 0.9 from not assigning homework or a payoff of 1 from assigning homework. He will therefore assign homework, and a subgame-perfect Nash equilibrium is (H, G, G).

- *The student and TA play* (B, C). The professor's view of subsequent actions is now changed. He still can obtain 0.9 from not obtaining homework, but if he does assign homework the student will not put forth good effort and the TA will not grade it. This leads to his payoff from assigning homework being -1. Then he will not assign homework, and a subgame-perfect Nash equilibrium is (N, B, C).

So we have now obtained two pure-strategy subgame-perfect Nash equilibria: (H, G, G) and (N, B, C). Notice that the strategy of the TA is no longer contingent: since he and the student move simultaneously (effectively, since the TA does not know how much effort the student has put forth) there is no way that his move can be contingent on hers[1]!

Now, we are not quite done. Although this will not be asked of you in this class, we should notice that there may be a mixed-strategy Nash equilibrium in the student and TA's subgame; if there is, there will be a subgame-perfect Nash equilibrium in which the later two players play this mixed strategy. We'll go ahead and characterize this, mostly as an exercise in computing mixed-strategy Nash equilibrium and in looking at how the professor considers his choices.

To compute mixed-strategy Nash equilibrium, we proceed by setting up indifference conditions for the relevant players, then working forward algebraically to see what the mixing probabilities must be to support

[1]Notice that we have been ignoring the fact that both the student and the TA have strategies contingent on the professor's strategy: if the professor assigns homework, they follow the strategies indicated in our derivation of subgame-perfect Nash equilibrium; otherwise, they don't play so their strategies don't matter. In this sense, their actions are still contingent, but since they don't actually move when the professor does not assign homework there isn't really anything to specify. So we short the notation slightly to keep things intuitive, and avoid specifying what the student and TA do when the professor does not assign homework.

this indifference.

$$\begin{aligned}
& E\left[u_S\left(G,\sigma_T\right)\right] = E\left[u_S\left(B,\sigma_T\right)\right] \\
\iff & \Pr\left(s_T = G\right)(1) + \Pr\left(s_T = C\right)(-1) = \Pr\left(s_T = G\right)(-1) + \Pr\left(s_T = C\right)(0) \\
\iff & \sigma_T^G - \sigma_T^C = -\sigma_T^G \\
\iff & 2\sigma_T^G = \sigma_T^C \\
\iff & 2\sigma_T^G = 1 - \sigma_T^G \\
\iff & \sigma_T^G = \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
& E\left[u_T\left(G,\sigma_S\right)\right] = E\left[u_T\left(C,\sigma_S\right)\right] \\
\iff & \Pr\left(s_S = G\right)(0) + \Pr\left(s_S = B\right)(-1) = \Pr\left(s_S = G\right)(-1) + \Pr\left(s_S = B\right)(1) \\
\iff & \sigma_S^G - \sigma_S^B = -\sigma_S^G \\
\iff & 2\sigma_S^G = \sigma_S^B \\
\iff & 2\sigma_S^G = 1 - \sigma_S^G \\
\iff & \sigma_S^G = \frac{1}{3}
\end{aligned}$$

From this, we see that $\sigma_S^G = \frac{1}{3}$, $\sigma_T^G = \frac{1}{3}$ (implying $\sigma_S^B = \frac{2}{3}$, $\sigma_T^C = \frac{2}{3}$) constitutes a mixed-strategy Nash equilibrium in the second-stage subgame. To complete this analysis, we need to see what the professor would like to do if the student and the TA follow these mixed strategies.

We still know that if the professor does not assign homework, he receives a payoff of 0.9; what remains is to compute his utility from assigning homework. However, since the actions of the student and TA are no longer deterministic, we cannot compute an exact utility. We must then compute his expected utility from each action and see which is larger. Since not assigning homework obtains a payoff of 0.9 with certainty, the expected payoff of not assigning homework is also 0.9. Unfortunately, some math will be involved in the second case.

If the professor does assign homework, we see that his expected utility is

$$\begin{aligned}
E\left[u_P\left(H,\sigma_S,\sigma_T\right)\right] = {} & \Pr\left(s_S = G, s_T = G\right)(1) + \Pr\left(s_S = G, s_T = C\right)(0)\ldots \\
& + \Pr\left(s_S = B, s_T = G\right)(-1) + \Pr\left(s_S = B, s_T = B\right)(-1)
\end{aligned}$$

Although the student and the TA are both randomizing in the second stage, they are doing so independently. Harking back to Econ 41, we know that the probability of two independent events occurring is the product of the probability that either one of them occurs. That is,

$$\begin{aligned}
& \Pr\left(s_S = G, s_T = G\right)(1) + \Pr\left(s_S = G, s_T = C\right)(0)\ldots \\
& + \Pr\left(s_S = B, s_T = G\right)(-1) + \Pr\left(s_S = B, s_T = B\right)(-1) \\
= {} & \Pr\left(s_S = G\right)\Pr\left(s_T = G\right)(1) + \Pr\left(s_S = G\right)\Pr\left(s_T = C\right)(0)\ldots \\
& + \Pr\left(s_S = B\right)\Pr\left(s_T = G\right)(-1) + \Pr\left(s_S = B\right)\Pr\left(s_T = C\right)(-1)
\end{aligned}$$

According to the mixed strategies derived above, this will become

$$\begin{aligned}
\Pr\left(s_S = G\right)\Pr\left(s_T = G\right)(1) + \Pr\left(s_S = G\right)\Pr\left(s_T = C\right)(0)\ldots & \\
+ \Pr\left(s_S = B\right)\Pr\left(s_T = G\right)(-1) + \Pr\left(s_S = B\right)\Pr\left(s_T = C\right)(-1) &= \sigma_S^G\sigma_T^G - \sigma_S^B\sigma_T^G - \sigma_S^B\sigma_T^C \\
&= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) \\
E\left[u_P\left(H,\sigma_S,\sigma_T\right)\right] &= -\frac{5}{9}
\end{aligned}$$

Then the professors sees and expected utility of $-\frac{5}{9}$ from assigning homework. Since his expected utility from not assigning homework is 0.9, his choice is obvious: don't assign homework. Then in the game with incomplete information for the TA, we find a third subgame-perfect Nash equilibrium,

$$\left(N, \sigma_S^G = \frac{1}{3}, \sigma_T^G = \frac{1}{3}\right)$$

Thus the game with incomplete information has three subgame-perfect Nash equilibria: two in pure strategies and one in which the student and the TA mix.

It is only fair to point out that the second and third parts of this question (the two parts with incomplete information on the part of the TA) are not going to be covered in this quarter of Econ 101. With that in mind, don't be too concerned if you're having trouble figuring out what's going on. Still, since we technically have all the tools we need to solve them we might as well push the boundaries.