Our standard solution concept

We are by now familiar with finding a consumer's optimal demand by setting

$$\frac{\text{MU}_x}{p_x} = \frac{\text{MU}_y}{p_y}.$$

So long as we do not have a corner solution (and utility is well-behaved), this equation will give us a useful expression for x in terms of y — or vice-versa — which we may then substitute back into the budget constraint to obtain a solution.

For the purposes of analyzing when and how corner solutions arise, it is important to understand exactly what this equation is saying: we have been phrasing it as, "equal marginal utility per unit cost," but intuitively what it is saying is that if I give up a little bit of x and substitute toward y, I have no gain in utility.

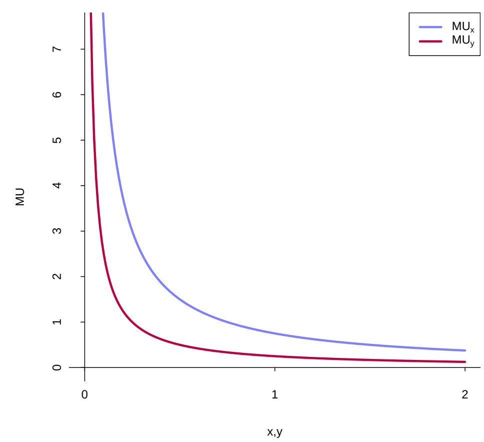

Figure 1: graphs of marginal utilites for x and y.

Consider the Cobb-Douglas case of $u(x, y) = \alpha \ln x + (1-\alpha) \ln y$, and let $p_x = 1$, $p_y = 2$. We can see

$$\text{MU}_x = \frac{\alpha}{x}, \qquad \text{MU}_y = \frac{1-\alpha}{y}.$$

The marginal utility equation then says that

$$\frac{\text{MU}_x}{p_x} = \frac{\text{MU}_y}{p_y} \quad \rightsquigarrow \quad \frac{\alpha}{x} = \frac{1-\alpha}{2y} \quad \implies \quad x^* = 2\left(\frac{\alpha}{1-\alpha}\right) y^*.$$

Now imagine that I am consuming $x' = 2\alpha y/(1-\alpha) - \varepsilon$, where $\varepsilon > 0$; that is, I am consuming something less than my optimal amount of x relative to y. With this ε in x I give up, I can afford $\varepsilon/2$ units of y; that is, since $p_x = 1$, by sacrificing ε of x I gain ε of purchasing power, which I can use to purchase y at a price of 2, hence I gain $\varepsilon/2$ units of y.

We then have $x' = x^* - \varepsilon$ and $y' = y^* + \varepsilon/2$. At this point, we have

$$\text{MU}_x = \frac{\alpha}{x^* - \varepsilon}, \qquad \text{MU}_y = \frac{1-\alpha}{y^* + \frac{\varepsilon}{2}};$$

in the marginal utility equation, this is

$$\left.\frac{\text{MU}_x}{p_x}\right|_{x'} = \frac{\alpha}{x^* - \varepsilon} > \frac{\alpha}{x^*} = \frac{1-\alpha}{2y^*} > \frac{1-\alpha}{2\left(y^* + \frac{\varepsilon}{2}\right)} = \left.\frac{\text{MU}_y}{p_y}\right|_{y'}.$$

That is, by sacrificing a little bit of y I lose less utility than I can buy back by reinvesting this sale in x. Since I am looking to maximize utility, this should definitely be done!

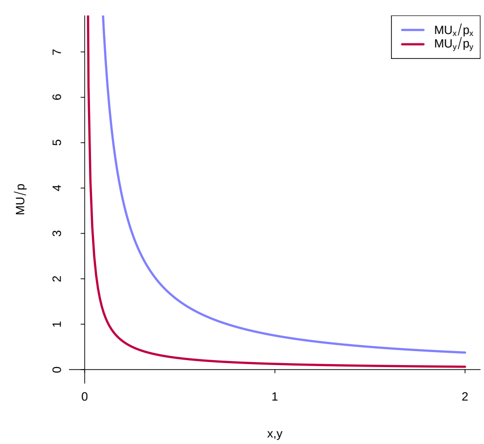

Figure 2: graphs of marginal utility per price ratios for x and y.

Graphically

The above is essentially a recap of what we've discussed this quarter. However, it is helpful for the purposes of intuition to see a graphical representation of the above balancing act. We'll continue the above example, under the additional assumption that $\alpha = 3/4$; then marginal utilities are given by

$$\text{MU}_x = \frac{3/4}{x}, \qquad \text{MU}_y = \frac{1/4}{y}.$$

These functions are shown in Figure 1.

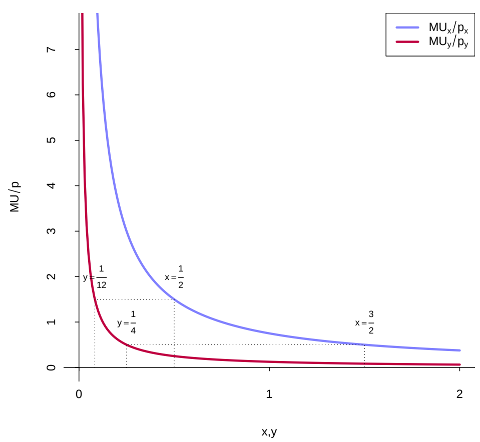

Figure 3: illustration of marginal utility per price ratios being equal between x and y.

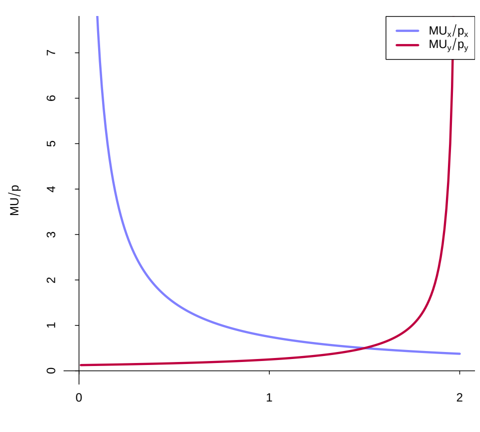

Figure 4: marginal utility per price graphed, where y is no longer an independent variable but is instead the result of spending all residual income after purchase of x.

Of course, we are not interested in marginal utility, but in its ratio to price. Accounting for $p_x = 1$ and $p_y = 2$, we have

$$\frac{\text{MU}_x}{p_x} = \frac{3}{4x}, \qquad \frac{\text{MU}_y}{p_y} = \frac{1}{8y},$$

These functions are shown in Figure 2.

We know that we should look for locations where the price ratios are equal; when we solve through the above, these solutions have $x = 6y$. Then when $y = 1/6$, $x = 1$, and when $y = 1/4$, $x = 3/2$. These points are plotted in Figure 3.

Admittedly, though, this figure is difficult to interpret: you have to look for identical elevations on the vertical axis, then measure off the x and y corresponding to these elevations. This does not do a very good job of capturing the tradeoffs inherent in optimization (i.e., more x implies less y, and vice-versa).

In order to consider tradeoffs, we will need to consider the budget constraint $p_x x + p_y y = w$; that is, without a budget constraint, there aren't tradeoffs to be made, and the agent would consume everything! Let's assume that the agent has wealth $w = 2$, so that the budget constraint is

$$p_x x + p_y y = w \quad \rightsquigarrow \quad x + 2y = 2.$$

A clear implication of this is that $y = 1 - x/2$.[1] So now, instead of plotting MU_x/p_x and MU_y/p_y on the same graph, we can plot MU_x/p_x and what MU_y/p_y *would be* if it were known that $y = 1 - x/2$. That is, our functions become

$$\frac{\text{MU}_x}{p_x} = \frac{3}{4x}, \qquad \frac{\text{MU}_y}{p_y} = \frac{1}{8(1 - x/2)} = \frac{1}{8 - 4x}.$$

These functions are shown in Figure 4.

Reading Figure 4, for a given x on the horizontal axis there is an implied $y = 1 - x/2$. We can see that when x is small, MU_x/p_x is large as we would expect; however, when x is small, MU_y/p_y is also small: this is because when x is small y is (relatively) large, and when y is relatively large it should be that MU_y/p_y is small.

To equate marginal utility per price is simple in this graph: by using the budget constraint to determine y as a function of x, we know that each point on the horizontal axis represents a feasible level of consumption. When x is small, we know that MU_x/p_x is larger than MU_y/p_y and when x is large, we know that MU_y/p_y is larger than

[1] In particular, since Walras' law says, under fairly general assumptions, that all wealth is spent on commodities we can always establish a relationship like this between x and y.

MU_x/p_x. Since we are looking for a point where these ratios are equal, the consumer is maximizing utility exactly where these lines cross; in particular, this occurs at $x = 3/2$, implying $y = 1 - (3/2)/2 = 1/4$. Note that this point was shown earlier — implying that it might be optimal — but absent the budget constraint it was impossible to know whether or not it was the solution to demand at this particular level of wealth, $w = 2$.

Corner solutions

We have thought so far about corner solutions as arising when we would like optimal consumption of one good to be negative. This is a particular case of a more general concept: consumption bundles must be *feasible*. That is, there must be no outside rules stipulating that the chosen allocation cannot arise. It is simple and intuitive to consider the rule that consumption must be weakly positive — after all, what is a negative apple? — but there is much more power than this specific feature. In the more general context, "corner" refers to being at the edge of what is feasible, not necessarily where the budget frontier hits the axis.

As an example, consider a law that you cannot own more than five firearms. You may have sufficient income to afford a much larger arsenal and your preferences may indicate that you should do so; however, by law you are constrained to own no more than five. At this point, it is no longer necessary that your marginal utility per price for firearms equals that for all other goods: you would own more guns if you could! Technically speaking, this is still a corner allocation/solution, although this is less graphically intuitive.

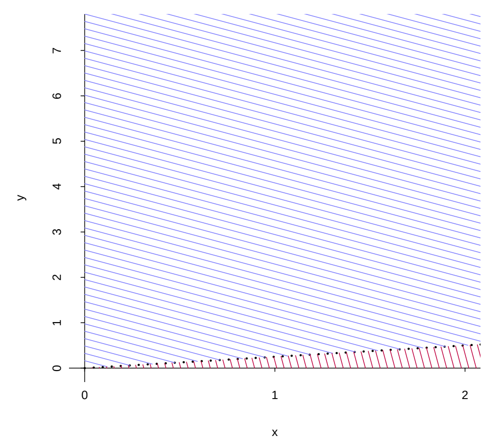

Figure 5: the feasible set (blue) and infeasible set (red) under government regulation; notice that $x \leq 4y$ is the same as $y \geq x/4$.

Regarding the above example, since utility is Cobb-Douglas[2] we know that there are no "proper" corner solutions along the axes.[3] However, we can still artificially generate fishy behavior.

Suppose that the government enters the picture — for whatever reason — and says that you cannot consume more than 4 units of x for every unit of y. The set of allocations which is legally feasible is shown in Figure 5.

This can be overlaid on Figure 4 after a little bit of manipulation. Remember that, in addition to $x \leq 4y$, we also know from the budget constraint that $y = 1 - x/2$. Hence

$$x \leq 4y = 4\left(1 - \frac{x}{2}\right) = 4 - 2x \quad \Longrightarrow \quad x \leq \frac{4}{3}.$$

Thus any point where $x > 4/3$ is not feasible when we take into consideration both the law and the budget constraint. We capture feasibility together with tradeoffs in Figure 6.

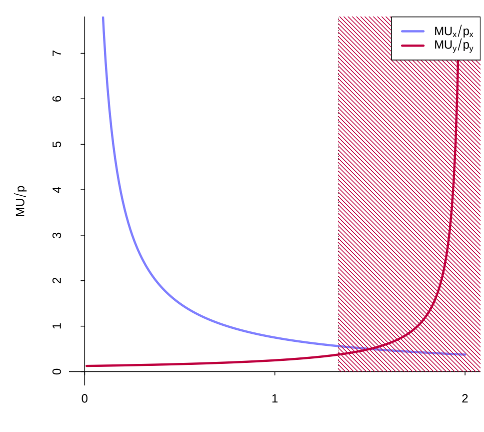

Figure 6: budget-constrained tradeoffs with infeasible allocations (red hashes) marked off.

Importantly, within the set of feasible allocations and taking into account the budget constraint, it is always the case that

[2] If this is unclear, remember that an increasing transformation of a utility function will represent the same preferences. Exponentiate the above utility function — $e^{u(x,y)}$ — and see what comes back.

[3] See the week 2 (?) notes regarding the Inada conditions.

$\text{MU}_x/p_x > \text{MU}_y/p_y$. Thus at every feasible allocation the consumer would like to trade some y for some x (as discussed above); hence she will consume as much x as is allowed, which we have determined is $x = 4/3$. This implies $y = 1 - (4/3)/2 = 1/3$.

Using this approach, how a corner solution arises is fairly apparent; we simply need to keep in mind both the inherent tradeoffs of budget-constrained optimization and the constraint that consumption must be feasible (in this case, not banned by the government). Since the agent is always looking to get better bang for her buck she will consume at what we term her "upper bound."

In a sense, the standard notion of a corner solution that we've been dealing with so far is exactly like this. Rather than the government passing a law that $x \leq 4y$, reality imposes a constraint that $x \geq 0$ and $y \geq 0$. When we block off the infeasible set in this case, we are simply blocking off all negative allocations; but this is what we do by only drawing the positive quadrant anyway! That is to say, the fact that we only plot positive x and y when we draw budget sets and indifference curves comes from the fact that consumption must be weakly positive. We could, of course, plot negative bundles and then explain that we're ignoring them, but since this is never *not* the case it is easier to use graphical shorthand and ignore all weakly-negative quadrants.

The takeaway here is that the marginal utility equation is useful, but we always need to keep in mind that consumption must be feasible according to the rules of the world (positivity, laws, etc.). In many cases this is not an issue, but be alert for curveballs coming your way. When feasibility becomes an issue, allow your intuition to take over: if the marginal utility per price is uniformly greater for one good than another within the whole of the feasible set, this good should be the only good purchased at the optimum.